MATHEMATICS MAGAZINE

CONTENTS

VOLUME 35 • SEPTEMBER 1962 • NUMBER 4

MATHEMATICS MAGAZINE

ROBERT E. HORTON, *Editor*

EDITORIAL CORRESPONDENCE should be sent to the Editor, ROBERT E. HORTON, Department of Mathematics, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Buffalo, New York, bi-monthly except July-August. Ordinary subscriptions are: 1 year \$3.00; 2 years \$5.75; 3 years \$8.50; 4 years \$11.00; 5 years \$13.00. Members of the Mathematical Association of America may subscribe at the special rate of 2 years for \$5.00. Single copies are 65¢.

PUBLISHED WITH THE ASSISTANCE OF THE JACOB HOUCK MEMORIAL FUND OF THE MATHEMATICAL ASSOCIATION OF AMERICA

Second class postage paid at Buffalo, New York and additional mailing offices.

STATISTICAL DECISION PROCEDURES IN INDUSTRY

II. CONTROL CHARTS BY ATTRIBUTES

BEN K. GOLD, Los Angeles City College

2.1. Introduction. The first article in this series, which appeared in the May 1962 issue of this magazine, discussed some of the philosophy and techniques involved in the use of control charts as an aid in making decisions about the quality of product being turned out by a manufacturing process when the quality characteristic under consideration is a measurable quantity. Oftentimes, considerations of practicality, convenience or economics rule out the use of this type of control chart.

The following situations illustrate some of these considerations:

1) The variables quality characteristic is impossible or difficult to measure. Examples—spongy iron, cracked castings, paper tears and wrinkles, metal rust or breakage, closure of electrical circuits, certain clinical procedures.
2) Measurements cannot be justified economically. Examples—stampings, screw threads or any situation where the cost of measurement is too high relative to the cost of production.
3) More than one dimension on a part is to be checked and an overall control of quality is desired.

In this article, we shall consider control charts for use when inspection is accomplished by the method of attributes, i.e., the quality characteristic is not measured but is simply classed as acceptable or non-acceptable.

For additional illustrations of situations leading to attribute control chart use, the reader is referred to [2.1] or [2.4].

The basic statistics used in attributes control charts are the sample **fraction defective**, p, and the sample **number of defects per unit**, c. The sample fraction defective is used whenever a unit in its entirety is classified as defective or non-defective. The sample number of defects per unit is used when the entire unit is not judged as defective or non-defective, but defects that appear in or on the unit are counted. We define

$$p = \text{sample fraction defective} = \frac{\text{number of defective units in sample}}{\text{total number of units in sample}}$$

$$c = \text{sample number of defects per unit} = \frac{\text{total number of defects counted in sample}}{\text{total number of units in sample}}$$

Sometimes the sample percent defective, $100\,p$, is used in place of the sample fraction defective.

Associated with the two statistics p and c are the corresponding population parameters p' and c'.

2.2. Sampling distribution of p. When samples of size n are drawn from a large population of items classified simply as defective or non-defective, and when the fraction defective in the population is p', the distribution of sample p values follows the **binomial distribution**, in which the probability of a sample fraction defective p is given by

$$f(p) = \frac{n!}{(np)!(n-np)!}(p')^{np}(1-p')^{n-np}, \qquad p = \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \cdots, 1$$

The binomial distribution has been extensively tabulated in recent years. For example, see [2.6] or [2.7]. The sampling distribution of p for the case $n=50$, $p'=.04$, is illustrated in Fig. 2.1.

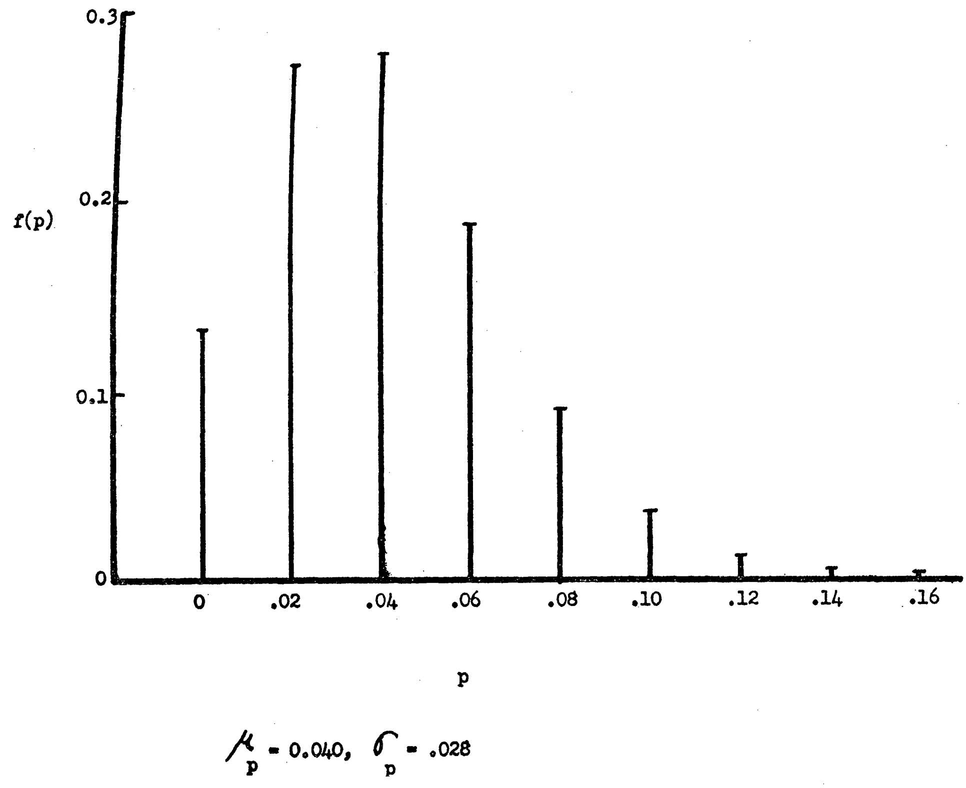


FIG. 2.1. Sampling distribution of p, $n=50$, $p'=0.04$.

Some properties of this sampling distribution, which is used in determining control limits for the p chart to be described in the next section, are:

1) It is not symmetrical unless $p'=0.50$.
2) It assumes the population is large (at least ten times the sample)
3) The average ($\bar{p}$) of sample p's gives the best estimate of p'.
4) The standard deviation of the distribution is given by

$$\sigma_p = \sqrt{\frac{p'(1-p')}{n}},$$

and approximately by

$$\sigma_p = \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

5) If $np' > 5$, the distribution approximates a normal distribution with

$$\mu_x = p', \qquad \sigma_x^2 = \frac{p'(1-p')}{n}$$

6) If $p' < 0.1$, the distribution approximates the Poisson distribution with $c' = np'$. The Poisson distribution will be discussed in Sec. 2.5.

2.3. The p Chart for Constant n. In order to illustrate the construction and use of a p chart, a box of beads was prepared containing a large number of white and red beads. A red bead was assumed to represent a defective article and a white bead an acceptable article. The data in Table 2.1 show the results of randomly drawing 20 samples of 50 each from this box in which the fraction of red beads was 0.04, i.e., $p' = 0.04$, $n = 50$. This technique can be thought of as simulating a month's run of a production process and a daily sample taken. The twenty p values are plotted consecutively in Fig. 2.2 as the preliminary data for a p chart. The center line and control limits are found from:

$$\text{UCL}_p = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}, \qquad \text{CL}_p = \bar{p}, \qquad \text{LCL}_p = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

If the calculated value for the lower control limit is negative, the value zero is used in its place. The quantity $\bar{p}$ is the *average sample fraction defective* and is found from:

$$\bar{p} = \frac{\text{total number of defectives found in all samples}}{\text{total number of pieces in all samples}}$$

If the p chart shows a state of statistical control, i.e., if no points are outside control limits and not more than seven consecutive points are on the same side of the center line, the process is assumed to be operating consistently or from chance cause variations only, and the control limits are extended. If a later point goes out of control, the presence of an assignable cause is indicated.

To illustrate the sensitivity of the p chart, drawings numbered 21 to 30 were made from a box of beads 9% defective, and drawings numbered 31 to 40 from a box of 2% defective. These data are presented in Table 2.2 and illustrated on the p chart in Fig. 2.2.

Some important points to note are the following:

1) If a point falls above the Upper Control Limit, the process is not in a state of statistical control because it has changed for the worse. Causes should be looked for and eliminated.
2) If a point falls below the Lower Control Limit, the process is not in a

state of statistical control because it has changed for the better. Causes should be looked for and incorporated into the process if possible.

3) If a p chart has a zero lower limit, it is impossible to detect an assignable cause for the better by means of a point being below the Lower Control Limit. It becomes most important then to observe any runs of points below the center line.
4) It should be noted that the above example is for a constant size sample. If the sample size varies, see Sec. 2.4.
5) Note the lack of sensitivity of the p chart as compared with measurements charts. In samples 21–30, the percent defect jumped from 4% to 9% but 7 of the ten points are within control limits.

TABLE 2.1

DATA FOR p CHART

Lot No.	n	d	p
1	50	2	.05
2	50	3	.06
3	50	1	.02
4	50	2	.04
5	50	2	.04
6	50	0	0
7	50	1	.02
8	50	1	.02
9	50	4	.08
10	50	1	.02
11	50	0	0
12	50	3	.06
13	50	1	.02
14	50	1	.02
15	50	0	0
16	50	0	0
17	50	5	.10
18	50	1	.02
19	50	4	.08
20	50	2	.04
Total	1000	34	

TABLE 2.2

DATA FOR p CHART EXTENDED

Lot No.	m	d	p
21	50	3	.06
22	50	4	.08
23	50	6	.12
24	50	9	.18
25	50	2	.04
26	50	6	.12
27	50	4	.08
28	50	3	.06
29	50	5	.10
30	50	4	.08
Total	500	46	$\bar{p} = .092$
31	50	0	0
32	50	1	.02
33	50	0	0
34	50	0	0
35	50	1	.02
36	50	0	0
37	50	3	.06
38	50	2	.04
39	50	1	.02
40	50	1	.02
Total	500	9	$\bar{p} = .018$

$$\bar{p} = \frac{34}{1000} = .034$$

$$= \sqrt{\frac{(.034)(.966)}{50}} = \sqrt{.000657} = .0256$$

$$3\sigma_p = .077$$

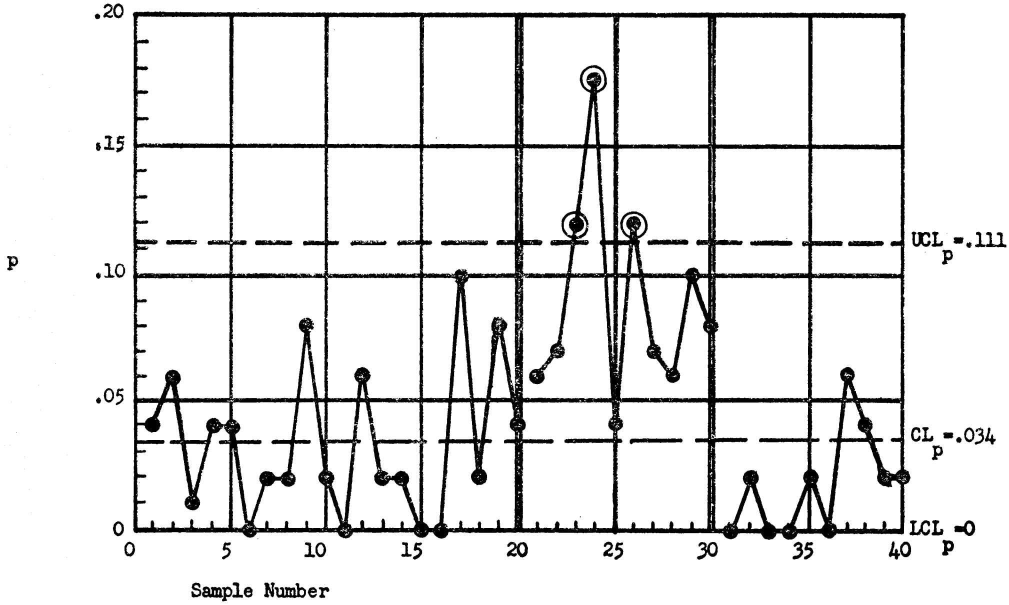


FIG. 2.2. p chart for data of Tables 2.1 and 2.2.

$$\text{UCL} = .034 + .077 = .111 \text{ or } 11.1\%$$
$$\text{LCL} = .034 - .077 = 0$$

If the sample size remains constant from sample to sample, the fraction defective is proportional to the number of defectives. Thus, the same chart can be calibrated along the vertical scale to read "number of defectives in sample (np)" rather than fraction defective. This permits the inspector to record only the number of defectives found, rather than calculate the fraction defective.

For the np chart:

$$\sigma_{np} = \sqrt{n\bar{p}(1 - \bar{p})} = n\sigma_p$$

$$\text{Center line} = n\bar{p}$$

$$\text{Upper Control Limit} = n\bar{p} + 3\sqrt{n\bar{p}(1 - \bar{p})}$$

$$\text{Lower Control Limit} = n\bar{p} - 3\sqrt{n\bar{p}(1 - \bar{p})}$$

The np chart is sometimes called the d chart.

Using the data in Table 2.1, we have the following results for the np chart illustrated in Fig. 2.3.

$$\text{UCL} = 1.7 + 3.8 = 5.5$$

$$\text{CL} = 1.7 \qquad \sigma_{np} = \sqrt{50(.0328)} = 1.28, \qquad 3\sigma_{np} = 3.8$$

$$\text{LCL} = 1.7 - 3.8 = 0$$

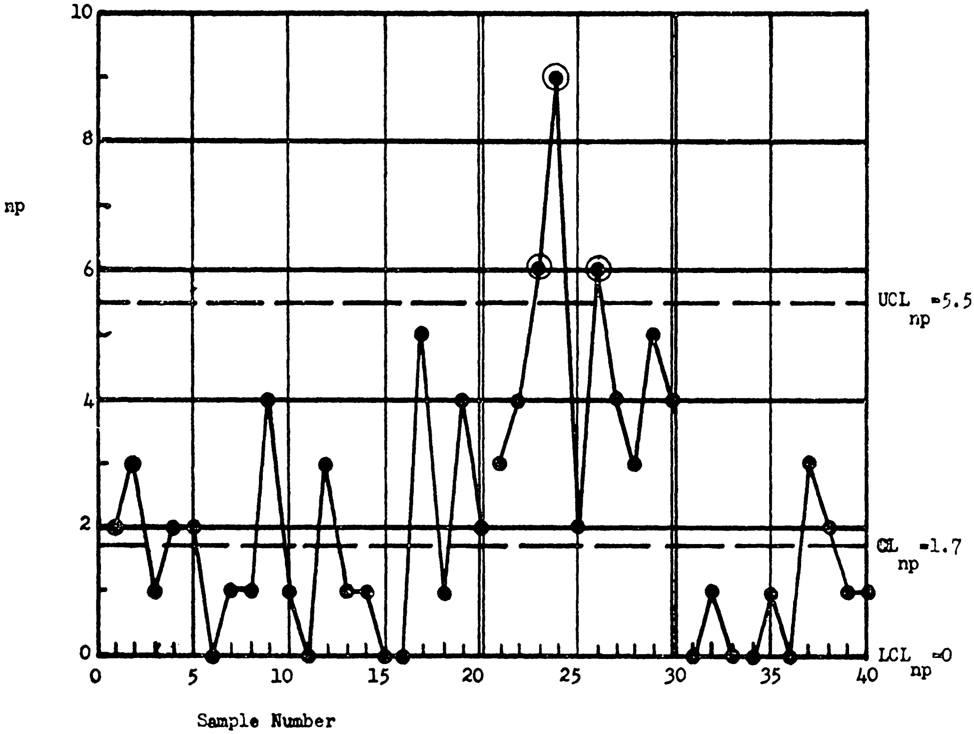


FIG. 2.3. np chart for data of Tables 2.1 and 2.2.

2.4. The p Chart for Non-Constant n. Often, in using a p chart, it is convenient to use as the sample size the total production of a unit of time such as an hour, a day, or a week. In this case, the sample size n will usually be non-constant. Since σ_p depends upon n, the control limits will vary from sample to sample. We shall indicate several common methods for handling this situation.

1) Control Limits for Each Point

Using the formulas

$$UCL_p = \bar{p} + 3\sigma_p, \qquad CL_p = \bar{p}, \qquad LCL_p = \bar{p} - 3\sigma_p,$$

control limits are figured separately for each point and drawn as shown in the example (Table 2.3 & Fig. 2.4).

It would be possible but quite awkward to use an np chart when the sample size varies, as the center line as well as the control limits would vary.

Calculation of Control Limits:

$$n = 50, \qquad .0407 \pm 3\sqrt{\frac{(.0407)(.9593)}{\sqrt{50}}} = .0407 \pm \frac{.5925}{7.07} = .0407 \pm .0838 = \begin{cases} .1245 \\ 0 \end{cases}$$

$$n = 100, \qquad .0407 + \frac{.5925}{\sqrt{100}} = .0407 \pm .0593 = \begin{cases} .1000 \\ 0 \end{cases}$$

TABLE 2.3

DATA FOR p CHART VARIABLE SAMPLE SIZE

Lot No.	No. Inspected	No. of Defectives	Fraction Defective	Control Limits	
				Lower	Upper
1	100	3	.030	0	.1000
2	50	5	.100	0	.1245
3	200	8	.040	0	.0826
4	150	9	.060	0	.0891
5	50	2	.040	0	.1245
6	250	9	.036	.0032	.0782
7	150	5	.033	0	.0891
8	100	4	.040	0	.1000
9	200	7	.035	0	.0826
10	250	11	.044	.0032	.0782
11	50	1	.020	0	.1245
12	150	9	.060	0	.0891
13	200	5	.025	0	.0826
14	100	4	.040	0	.1000
15	250	11	.044	.0032	.0782
16	50	2	.040	0	.1245
17	150	4	.027	0	.0891
18	200	8	.040	0	.0826
19	250	12	.048	.0032	.0782
20	100	3	.030	0	.1000
Totals	3000	122			

$$\bar{p} = \frac{122}{3000} = .0407$$

$$n = 150, \qquad .0407 \pm \frac{.5925}{\sqrt{150}} = .0407 \pm .0484 = \begin{cases} .0891 \\ 0 \end{cases}$$

$$n = 200, \qquad .0407 \pm \frac{.5925}{\sqrt{200}} = .0407 \pm .0419 = \begin{cases} .0826 \\ 0 \end{cases}$$

$$n = 250 \qquad .0407 \pm \frac{.5925}{\sqrt{250}} = .0407 \pm .0375 = \begin{cases} .0782 \\ .0032 \end{cases}$$

2) Control Limits Figured for Certain Sample Sizes Only

In this method, several control limits are drawn for several appropriate sample sizes, and each point is then checked against the proper control limit. If a point is close to its proper control limit, the control limits for the exact value of sample size should be checked.

A common modification of this method is to calculate one set of control limits, using the average sample size of n. Sample sizes which differ from the average more than about one fourth of the average should be checked individually, as well as any points that are very close to the limit.

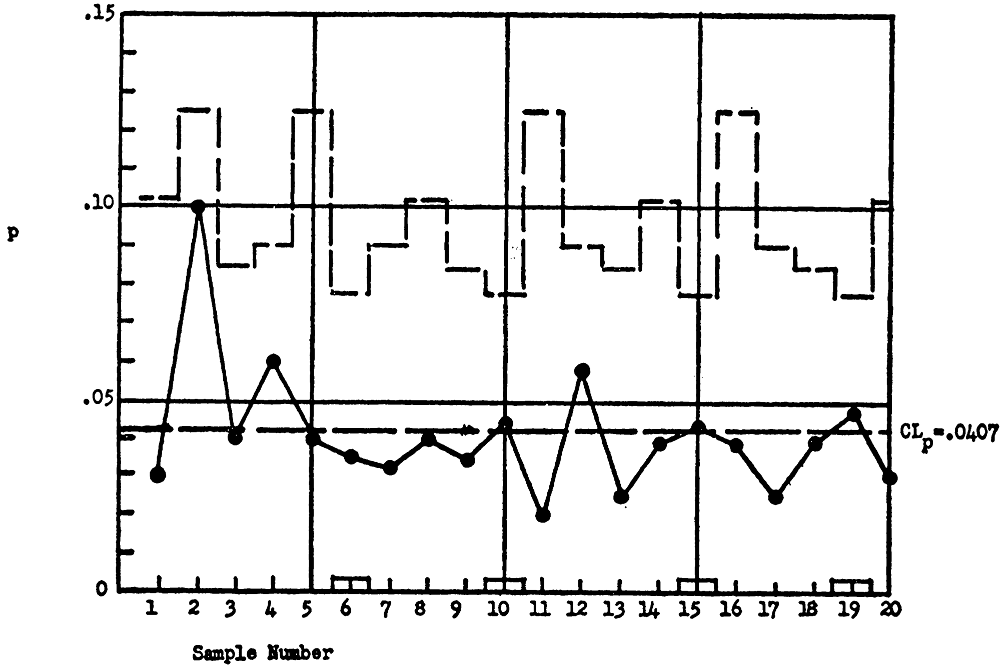


FIG. 2.4. p chart for data of Table 2.3.

3) The "Stabilized p" or t Chart

Another method of taking care of the variation in sample size is to leave the limits fixed and adjust the plotting of each point with respect to those limits. This is accomplished by standardizing the p values very much as normal curve values are standardized. We use the formula:

$$t = \frac{p - \bar{p}}{\sigma_p}.$$

When t is used the control limits become:

$$UCL_t = 3, \quad CL_t = 0, \qquad LCL_t = -3$$

The t values are then plotted on the chart instead of the p values.

The t chart has the advantage that it can be prepared ahead with the center line and the control limits printed in advance, but it has the disadvantage that one must calculate a t value for each point.

The data of Table 2.4 have been presented in Fig. 2.5 and Fig. 2.6 to illustrate the above two methods.

TABLE 2.4

DATA FOR p CHART (OR t CHART)

Sample No.	n Number Inspected	d No. of Defectives	p Fraction Defective	Columns for t chart		
				$p-\bar{p}$	σ_p	$t=\frac{p-\bar{p}}{\sigma_p}$
1	1450	42	.0290	−.0135	.0053	−2.55
2	1775	65	.0368	−.0057	.0048	−1.19
3	842	27	.0321	−.0104	.0070	−1.49
4	1590	75	.0472	.0047	.0051	0.92
5	1017	43	.0432	−.0002	.0063	−0.03
6	2345	84	.0343	−.0082	.0042	−1.95
7	950	55	.0579	.0154	.0066	2.34
8	1217	83	.0682	.0257	.0058	4.43
9	1432	92	.0642	.0217	.0053	4.10
10	1875	112	.0605	.0180	.0047	3.83
11	2510	115	.0458	.0033	.0040	0.83
12	1215	75	.0617	.0188	.0058	3.24
13	1712	87	.0508	.0083	.0049	1.69
14	915	42	.0459	.0034	.0067	0.51
15	1432	40	.0279	−.0146	.0053	−2.76
16	1229	38	.0310	−.0115	.0058	−1.98
17	1725	49	.0284	−.0141	.0049	−2.98
18	1142	27	.0236	−.0189	.0060	−3.15
19	1319	35	.0268	−.0157	.0056	−2.80
20	1248	39	.0312	−.0113	.0057	−1.98
Totals	28,940	1225				

$$\bar{p} = \frac{1225}{28940} = .0425$$

Control Limits

n	Lower	Upper
1000	.0233	.0617
1500	.0268	.0582
2000	.0289	.0561
2500	.0304	.0564

Check for point #18

$$\sigma_p = \sqrt{\frac{(.0425)(.9575)}{1142}} = .00598$$

$$\text{LCL} = .0425 - 3(.00598) = .0425 - .0179 = .0243$$

Point is out of control

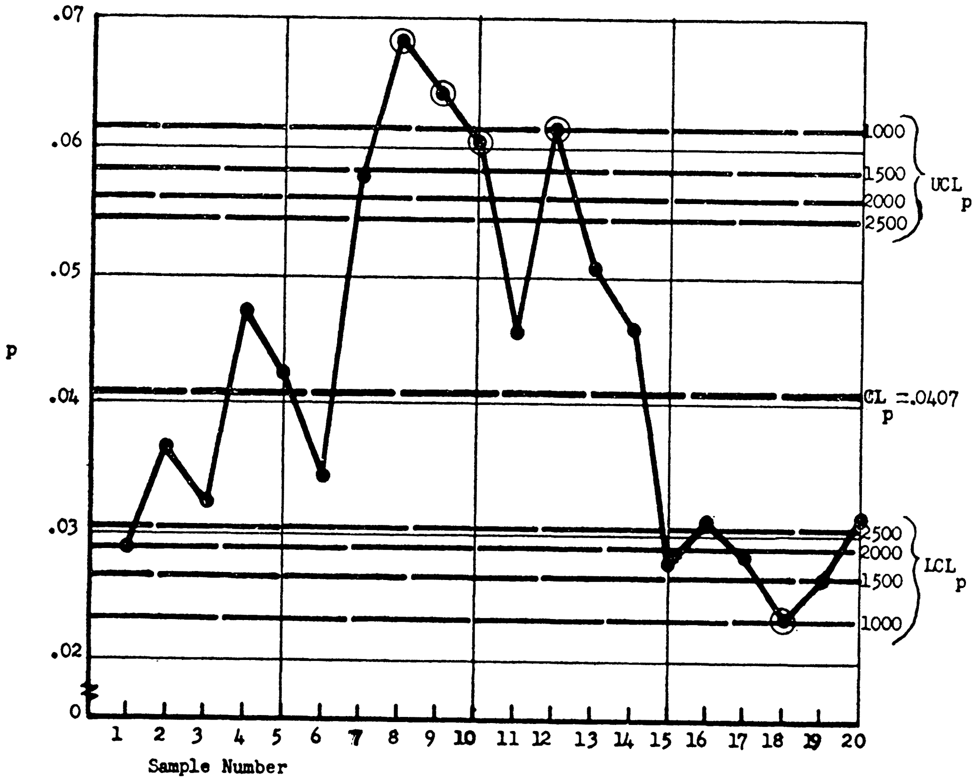


FIG. 2.5. p chart for data of Table 2.4.

2.5. Sampling Distribution of c. The theoretical model that is called upon when a c chart is used is the Poisson probability distribution function:

$$f(c) = \frac{(c')^c}{c!} e^{-c'}, \qquad c = 0, 1, 2, 3, \cdots \text{ to infinity}$$

where c' is the average number of defects per unit in the population and c is the number of defects in a sample of size one. For tabulated values of the Poisson distribution, see [2.5]. Fig. 2.7 illustrates the Poisson distribution for $c' = 2$.

It will be observed that the probability distribution functions in Fig. 2.1 and Fig. 2.7 are almost identical. The horizontal scales would be identical if in Fig. 2.1, the number of defectives (np) instead of the fraction defective (p) were plotted. In fact, the Poisson distribution is the limiting case of the binomial distribution as $n \to \infty$ and $p' \to 0$ in such a way that np' remains constant and equal to c'. The practical result of this consideration is that the Poisson distribution can be used as an approximation to the binomial distribution for small values of p', say $p' < 0.1$, and large values of n, say $n > 50$.

Some important properties of the Poisson distribution are:

1) It is non-symmetrical.
2) The average ($\bar{c}$) of sample c's gives the best estimate of c'.

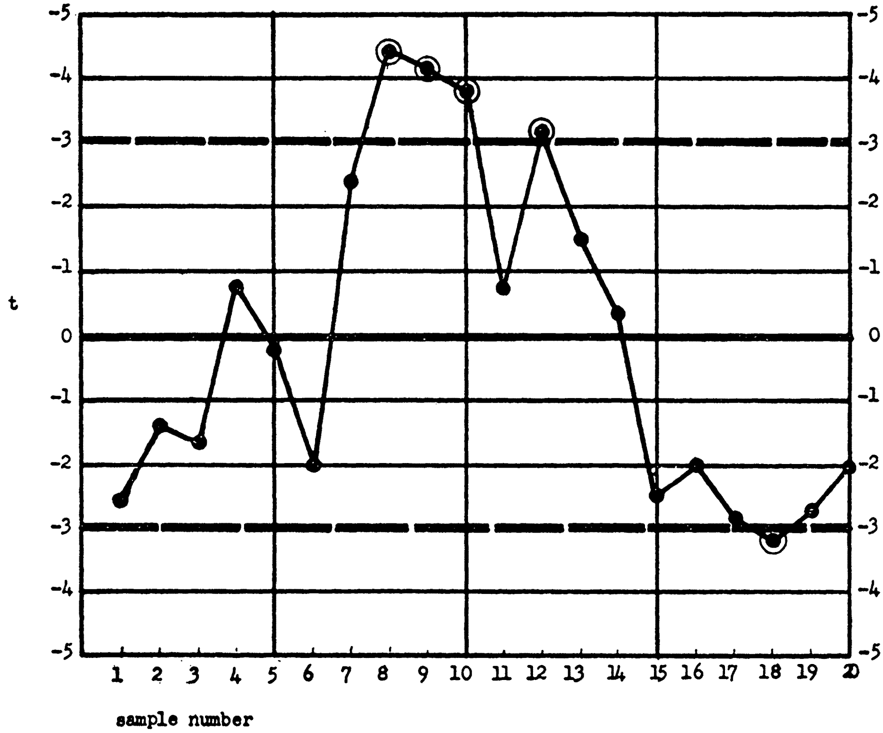


FIG. 2.6. t chart for data of Table 2.4.

3) The standard deviation of a Poisson distribution with average c' is given by $\sigma_c = \sqrt{c'}$.
4) The Poisson distribution approximates the binomial distribution for large n and small p'.
5) For large values of c', the Poisson distribution approximates the normal distribution with $\mu_c = c'$, $\sigma_c^2 = c'$.

2.6. The c Chart. To illustrate the construction of a c chart, drawings were made one at a time (with replacement) from decks of cards prepared to simulate conditions under which use of a c chart would probably be advisable.

Control limits for the c chart are obtained as follows:

$$UCL_c = \bar{c} + 3\sqrt{\bar{c}}, \qquad CL_c = \bar{c}, \qquad LCL_c = \bar{c} - 3\sqrt{\bar{c}}$$

Samples 1–20 in Table 2.5 were drawn from a distribution with $c'=8$. Samples 21–30 and 31–40 were drawn from distributions with $c'=16$ and $c'=4$ respectively. These data are plotted on a c chart in Fig. 2.8. Control limits were figured on the basis of the first twenty samples.

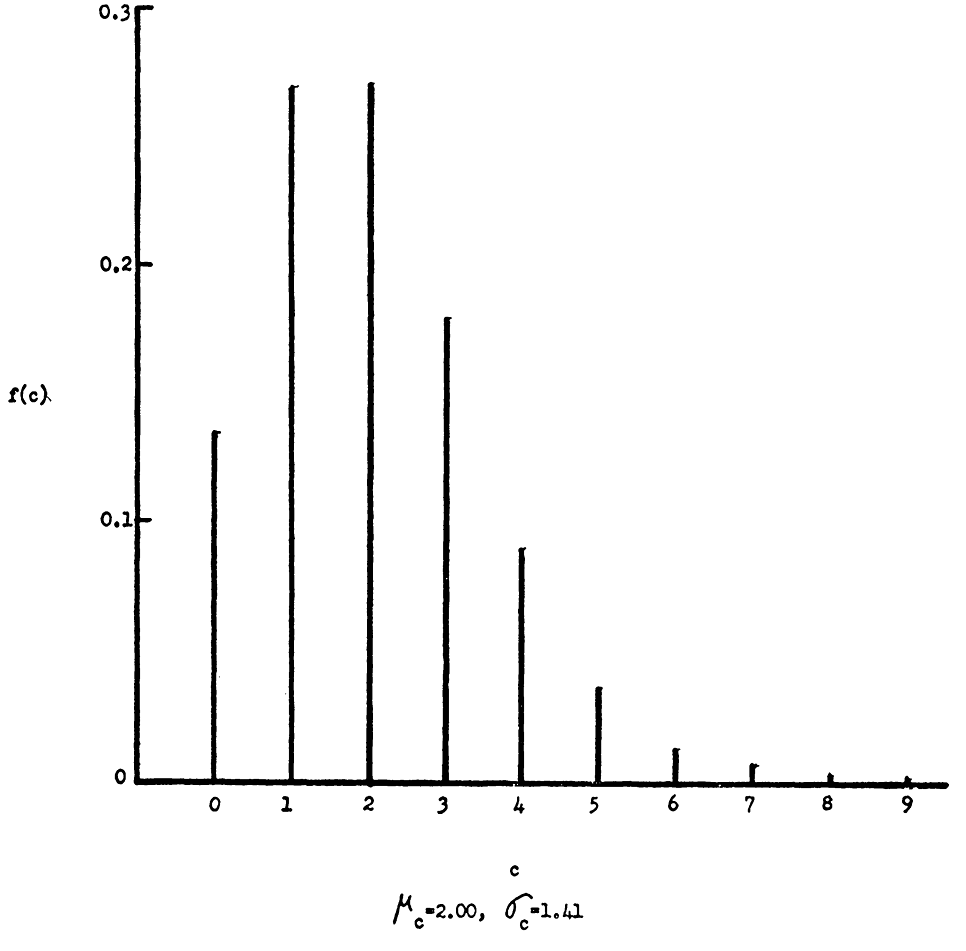


FIG. 2.7. Poisson distribution, $c'=2$.

TABLE 2.5

DATA FOR c CHART

Sample Number	Number (c) of Defects	Sample Number	Number (c) of Defects
1	13	21	23
2	4	22	14
3	7	23	17
4	3	24	16
5	8	25	14
6	4	26	17
7	9	27	17
8	11	28	21
9	10	29	11
10	8	30	19
11	6	31	5
12	10	32	2
13	5	33	5
14	7	34	4
15	5	35	4
16	10	36	7
17	9	37	5
18	6	38	4
19	5	39	5
20	13	40	4
Total	153		

$$\bar{c} = \frac{153}{20} = 7.65, \qquad \sqrt{\bar{c}} = \sqrt{7.65} = 2.766$$

$$\text{UCL} = 7.65 + 3(2.766) = 7.65 + 8.30 = 15.95, \qquad \text{LCL} = 0$$

2.7. The $\bar{c}$ Chart. If it is necessary or advisable to inspect k units for one sample, and this number varies, the $\bar{c}$ chart is useful, where $\bar{c}$ is the average number of defects per unit in the sample. As an average becomes more stable when the number of units is increased, the control limits will become closer together as the number of sample units inspected increases. For the $\bar{c}$ chart, we have:

$$\text{UCL}_{\bar{c}}\text{: } \bar{\bar{c}} + 3\sqrt{\frac{\bar{\bar{c}}}{k}} \qquad \text{CL}_{\bar{c}}\text{: } \bar{\bar{c}} \qquad \text{LCL}_{\bar{c}}\text{: } \bar{\bar{c}} - 3\sqrt{\frac{\bar{\bar{c}}}{k}}$$

where k is the number of units inspected in the sample, and

$$\bar{\bar{c}} \text{ is defined as } \frac{\text{total number of defects}}{\text{total number of units inspected}}$$

The situation here is comparable to that of the p chart for variable sample size.

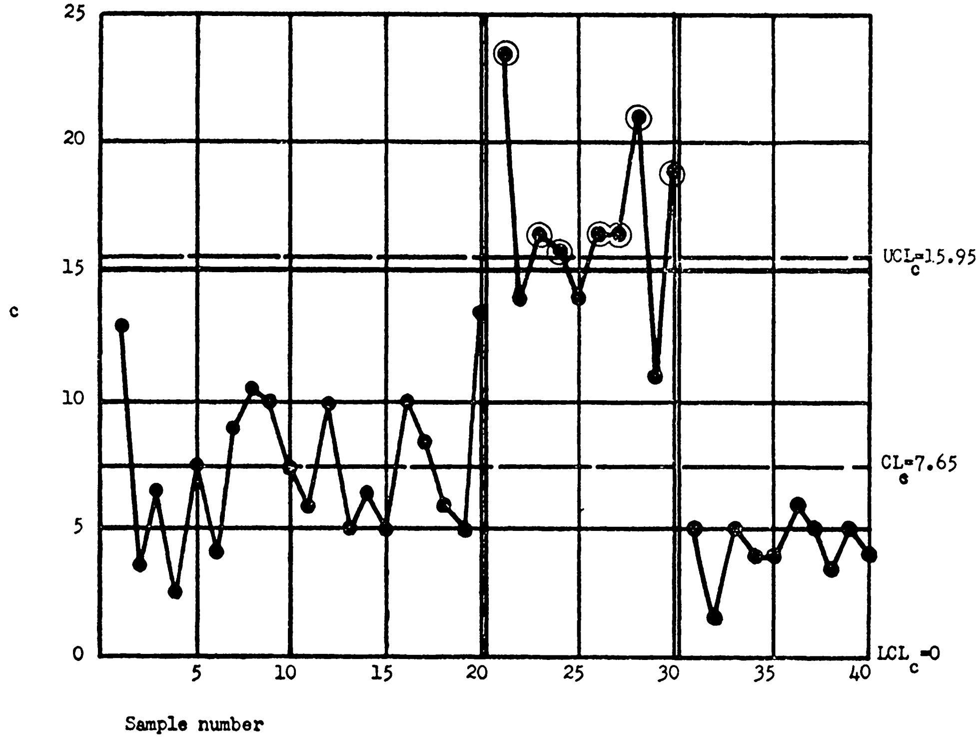


FIG. 2.8. c chart for data of Table 2.5.

2.8. Operating Characteristics for Control Charts. As discussed in Sec. 1.6, each time a point is plotted on a control chart after control limits have been extended from preliminary data, a decision is made as to the actual state of the production process. If the point is within the control limits, the decision made is to accept the hypothesis, H_0, that the production process is in an acceptable state of statistical control. If the point is not within the control limits, the decision is made to reject this hypothesis. In either case, there exists a possibility of an incorrect decision. The probability of deciding incorrectly that a process is not in an acceptable state of statistical control is fixed and can be determined in advance on the basis of certain assumptions about the distribution of the particular statistic involved. For the 3 sigma control chart, this so-called **Type I error** is usually about 0.003. The probability of deciding incorrectly that a process is in an acceptable state of statistical control varies, depending on the type and magnitude of the cause of lack of control—for example, a shift in μ_x or σ_x when an $\bar{x}$ chart is in use, or a shift in p' when a p chart is in use. The probability of making this so-called **Type II error** is thus a function of the parameter of the distribution. This function is called an *operating characteristic* of the control chart. Some operating characteristic curves for various sample sizes for a p chart based on a p' value of 0.02 are sketched in Fig. 2.9. For a more complete discussion of operating characteristic curves for control charts, the reader is referred to [2.2] or [2.3].

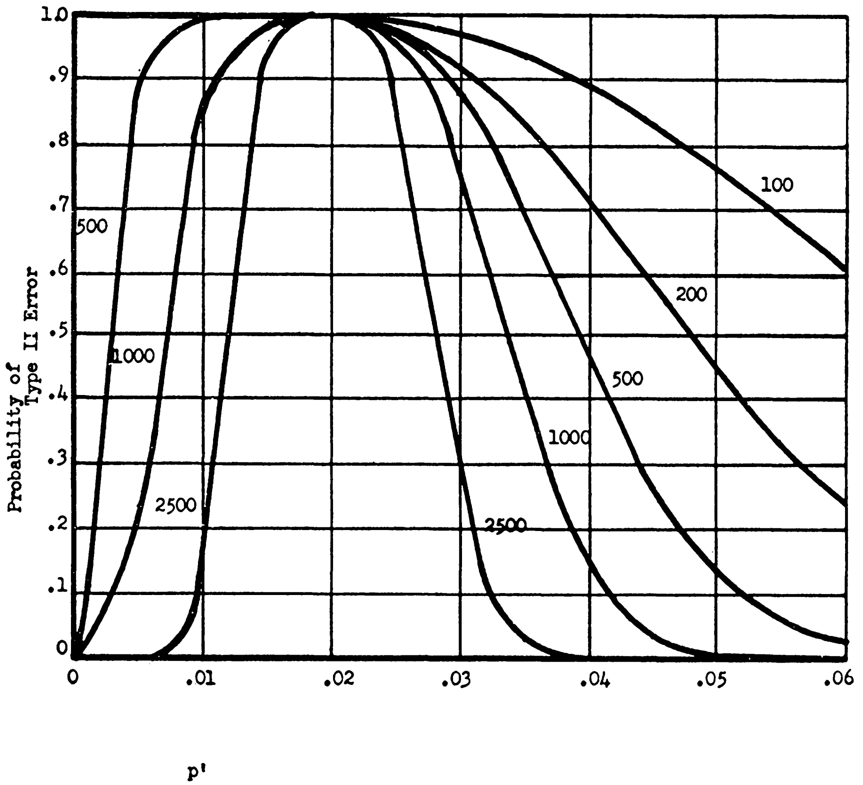


FIG. 2.9. Operating characteristic curves for p chart, based on $p' = 0.02$, various sample sizes.

2.9. Summary. These first two articles in this series of five have been concerned with the use of statistical techniques to make decisions about the quality of manufactured product while it is in the production stages. The next two articles will deal with the use of statistical techniques to make decisions about the quality of manufactured product as it passes from producer to consumer. The final article in the series will summarize the present state of statistical decision-making in industry with a look toward the future.

In closing the discussions concerning the control charts, let us consider some relative merits of variables and attributes charts:

1) variables charts are generally more sensitive to changes in the process.
2) variables charts generally require smaller sample sizes.
3) attributes charts require one chart; variables charts require two.
4) attributes charts are generally easier to read and interpret in terms of cost.
5) It is possible to combine several dimensions, processes, or even the work of a department on one attributes chart.

For detailed discussions of these and other comparisons between the two kinds of control charts, the reader is referred to [1.5], [1.6], [2.2] or [2.3].

2.10. Comments on Exercises Suggested in Sec. 1.8, Mathematics Magazine, May 1962, p. 141–142.

1) for the $\bar{x}$ chart, $CL_{\bar{x}} = -2.49$, $UCL_{\bar{x}} = 4.00$, $LCL_{\bar{x}} = -8.98$
   for the R chart, $CL_R = 6.48$, $UCL_R = 16.3$, $LCL_R = 0$.
2) for samples 26 to 40, no change in production process has taken place. All points on both charts should be within control limits except for rare chance factors.
3) for samples 41 to 50, average value of items has increased about 0.008″. This should be reflected on the $\bar{x}$ chart by several of the ten points lying above the UCL. The R chart remains in control.
4) for samples 51 to 60, the average value of the items remains at 2.000″ but the variation has about doubled. This is reflected by points above the UCL on the R chart, and possibly points out of control in either direction on the $\bar{x}$ chart due to the wider spread.
5) for samples 61 to 70, the average value has increased to about 2.010″ and the variation remains about doubled. Points above the UCL are now to to be expected frequently on both charts.
6) for samples 71 to 100, the average value is now about 2.003″ with the variation approximately that of the original 25 items. Thus the value for the $\bar{x}$ center line and control limits are about 3 units greater than the original, while the center line and control limits for R remain about the same.

References

2.1. I. Burr, J. Henry, L. Knowler and M. Wescott, Training Manual for the Two Day Intensive Training Course in Elementary Statistical Quality Control, Chicago Section, American Society for Quality Control, 1953.

2.2. Dudley J. Cowden, Statistical Methods in Quality Control, Prentice-Hall, Inc., 1957.

2.3. Acheson J. Duncan, Quality Control and Industrial Statistics, Revised Edition, Richard D. Irwin, Inc., Homewood, Illinois, 1959.

2.4. R. Freedman and J. Movshin, A Basic Training Manual on Statistical Quality Control. The St. Louis Society for Quality Control, 1951.

2.5. E. C. Molina, Poisson's Exponential Binomial Limit, D. Van Nostrand Co., Inc., 1959.

2.6. Harry G. Romig, 50–100 Binomial Tables, John Wiley and Sons, Inc., 1953.

2.7. Tables of the Binomial Probability Distribution, National Bureau of Standards, Applied Mathematics Series 6, 1950.

ON THE FAMILY OF CURVES $z=(t-t^n)/2$

SISTER M. STEPHANIE SLOYAN, Georgian Court College, Lakewood, New Jersey

This simple parametric equation

$$z = \frac{t - t^n}{2} \tag{1}$$

in complex coordinates describes an interesting family of curves whose loci are easily constructed and studied.

I. For n greater than 1, the curves are constructed in this manner: beginning with the unit point A mark off an arc AB in a positive direction on the unit circle, so that point B coincides with the unit vector t where $t=e^{i\theta}$ and θ is any desired amplitude. Beginning again with the unit point, lay off arc $AC=t^n$ ($t^n=e^{ni\theta}$ for real n), also in a positive direction. Find C', the point diametrically opposite C. The midpoint of chord BC' is a point of the desired locus.

Each curve of the family is of degree $2n$ and has $(n-1)$ points of tangency with the unit circle, at such points where $z=t=$any one of the $(n-1)$st roots of negative unity. For, given equation (1), points of tangency occur when $2t=t-t^n$, $t(t^{n-1}+1)=0$. Since t is a point on the unit circle, i.e., it is a unit vector, $t\neq 0$, and $t^{n-1}=-1$.

The origin is an $(n-1)$-tuple point of the curve. In this case, $0=z=(t-t^n)/2$, $t(t^{n-1}-1)=0$, $t\neq 0$, so $t^{n-1}=1$. The curve passes through the origin (pole) each time t takes on a value equal to one of the $n-1$ $(n-1)$st roots of unity.

Differentiating with respect to t, we find

$$dz/dt = \frac{1 - nt^{n-1}}{2} \cdot \qquad \text{Setting this equal to 0 yields}$$

$$t^{n-1} = 1/n. \tag{2}$$

Such values of t which lie on the unit circle, when substituted in (1) yield cusps of the curve. But, except in the cases where $n=\pm 1$ which will be discussed later, there is no t which will satisfy (2) and therefore no cusp.

The equation of the axis of reals is $z-\bar{z}=0$. Solving this with (1) gives:

$$t^{2n} - t^{n+1} + t^{n-1} - 1 = 0. \tag{3}$$

The axis of reals then may have $2n$ intersections with the curve.

(4) Equation (3) may be factored: $(t^{n-1} - 1)(t^{n+1} + 1) = 0$.

A. If n is even: The origin is an $(n-1)$-tuple point of the curve, there is one simple point of the curve on the axis of reals (where $z=t=-1$), and there are $n/2$ double points on the axis of reals, making a total of $2n$ intersections. The equation $t^{n-1}-1=0$ yields the values of t which when substituted in (1) give the origin. The equation $t^{n+1}+1=0$ yields values of t which do not lead to the origin. If n is even, then this last equation is of odd degree and always has the solution $t=-1$ which gives the point of tangency, and in addition has the factor $(t^n-t^{n-1}+\cdots+1)$. This has an even number of zeros which, in pairs, give the double points on the axis of reals. There are n such t's and therefore $n/2$ double points.

B. If n is odd: As above the equation $t^{n-1}-1=0$ yields values of t which give the origin. The equation $t^{n+1}+1=0$ yields values of t which give other points on the axis of reals. This last equation is always of even degree. If $n=3$, equation (4) is $(t^2-1)(t^4+1)=0$. The origin is a double point and there are two other double points on the axis of reals at a distance of $\pm\frac{1}{2}$ from the origin. If n is greater than 3 (and odd), the two equations $t^{n-1}-1=0$ and $t^{n+1}+1=0$ have in common the solutions $\pm i$. This reduces the number of intersections on the axis of reals and the curve to $2n-2$. The value $t=-1$ is not a root of either equation, and so there is no point of tangency there with the unit circle on the axis.

C. If n is greater than 1 and positive, the curve is symmetrical with respect to the axis of reals if n is even, and symmetrical both with respect to the axis of reals and to the axis of imaginaries if n is odd.

II. If n is negative and less than 1 the family may be written

$$z = \frac{t - 1/t^n}{2}. \tag{5}$$

Beginning with the unit point A, mark off an arc AB in a positive direction on the unit circle in the same manner as before and mark off AC so that $C=t^n$. Then by constructing a perpendicular to the axis of reals find the conjugate of C, $\overline{C}$, the image of C in the axis of reals. Determine $\overline{C}'$, the point diametrically opposite $\overline{C}$. The midpoint of the chord $B\overline{C}'$ is a point of the locus.

For $n=|2|$ or $|3|$ this construction yields the "roses" (rhodonea). For $|n|>3$, the "petals" overlap and there are double points. Each curve is tangent to the unit circle $|n|+1$ times, and the curve passes through the origin $|n|+1$ times. These curves also have no cusps. As in the case when n is positive, if n is even the curve is symmetrical with respect to the axis of reals, and symmetrical with respect to both axes if n is odd. Solving $z=(t-1/t^n)/2$ with the axis of real gives

$$t^{2n} + t^{n+1} - t^{n-1} - 1 = 0 \quad \text{which factors into}$$
$$(t^{n+1} - 1)(t^{n-1} + 1) = 0.$$

The $(|n|+1)$ $(|n|+1)$st roots of unit when substituted in (5) give the origin $|n|+1$ times.

A. If $|n|=2$, the curve is a three-leaved rose, with points of tangency where $z=t=$ the $|n|+1$ roots of -1. The origin counts as a triple point, and there is a simple point of the curve on the axis of reals. If $|n|$ is even and greater than $|2|$, $(t^{n+1}-1=0)$ gives the origin $(|n|+1)$ times. The equation $t^{n-1}+1=0$ always has the solution $t=-1$ and this yields a simple point of the curve. There are in addition $(|n|-2)/2$ double points on the axis of reals.

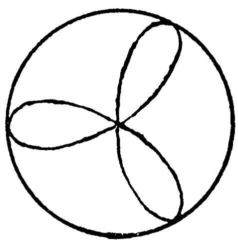
B. If $|n|$ is odd and $=3$, there is no overlapping. The equations of intersection with the axis of reals become: $(t^4-1)(t^2+1)=0$ and $\pm i$ are roots of both sections. Then there are $2|n|-2$ or four intersections (all at the origin) and no double points. If n is odd and greater three, the origin is an $(|n|+1)$-tuple point, and there are $(|n|-1)/2$ double points on the axis.

III. The degenerate cases are instructive.

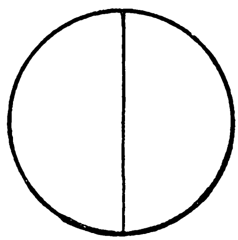
a. If $n=-1$ the curve becomes a degenerate two-leaved rose which is the same line as the axis of imaginaries. Equation (4) becomes $(t^2-1)(2)=0$, so that the origin is a double point of the curve and there are no other intersections with the axis of reals. There are two points of "tangency": $t=z=\pm i$. Thus this curve follows the same pattern as for other negative n.

b. If $n=0$, the curve is a circle with center on the axis of reals one-half unit to the left of the origin, and the radius of the circle is $\frac{1}{2}$. It may be considered as a "one-leaved rose" having the origin as a simple point, and with one point of tangency with the unit circle where $z=t=-1$. This curve, too, is not unusual and follows the pattern for positive n.

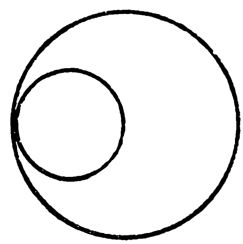
c. If $n=1$, however, the curve degenerates into a point at the origin, $z=0$. This is an isolated point, but by definition of cusp given before, might almost be considered one. Returning to equation (2), we find that if $n=1$, this equation is satisfied by any t, or by any point on the unit circle. Placing any of these t's in equation (1) with $n=0$ yields always $z=0$. Another way of regarding **III.c.** is this: If $n=-1$, then equation (2) has the two solutions $\pm i$. Placing these in equation (5) with $n=-1$ gives $\pm i$ again, but now they are cusps, and the figure can no longer be regarded as a two-leaved rose, but becomes a "hypocycloid of two cusps."



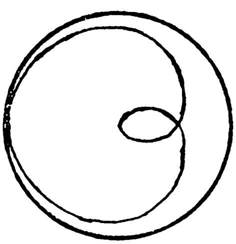
N = -2



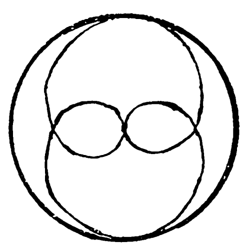
n = -1



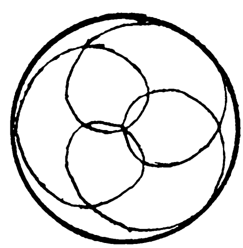
n = 0



n = 2



n = 3



n = 4

ON PARTICULAR PRODUCTS OF FUNCTIONS

GILBERT LABELLE, Collège de Longueuil, Canada

Certainly many persons learning differential calculus have made the following mistake in differentiating the product of functions $u(x)$ and $v(x)$. "The derivative of a product equals the product of the derivatives." In symbols:

$$(uv)' = u'v' \tag{1}$$

u and v not being constants. Let us find a particular relation between $u(x)$ and $v(x)$ by which equation (1) would be true.

Suppose that (1) is true. We have $u'v' = u'v + uv'$. With u' and v' not zero, that is, u and v not constants, we find that $u/u' + v/v' = 1$. Then

$$\frac{v}{v'} = \frac{u' - u}{u'} \quad \text{or} \quad \frac{v'}{v} = \frac{u'}{u' - u}$$

so upon integration

$$\ln v = \int \frac{du}{u' - u}\,.$$

Finally we have

$$v = \exp\left(\int \frac{du}{u' - u}\right). \tag{2}$$

We can make a table of some particular functions $u(x)$ and $v(x)$ satisfying (1)

u	x^n	e^{mx}	$\tan x$	etc.
v	$(n - x)^{-n}$	$e^{mx/(m-1)}$	$\exp\left(\frac{2}{\sqrt{3}} \operatorname{arc\,tan} \frac{2\tan x - 1}{\sqrt{3}}\right)$	etc.

Note that $u(x)$ and $v(x)$ may be multiplied by constants.

In conclusion, if one finds an exercise such as differentiating $x^n \cdot (n-x)^{-n}$, differentiate x^n then $(n-x)^{-n}$ and multiply the answers!

A "CONVERSE" TO FERMAT'S LAST THEOREM?

SAM SESSKIN, Hofstra College

The sum of the ordered positive integers $S_j = 1+2+3+ \cdots +j$ may be as simple and straightforward a matter as appears, but it also holds a wealth of hidden mathematical relationships to at least one topic apparently far removed.

The case in point arises from the question: When is S a square?

Perhaps, dealing with squares and integers as the problem does, we should suspect that S^2 would have an affinity for Pythagorean triangles, as it does. But the connection is hardly apparent. We offer three solutions to the problem; two are Pythagorean-oriented, and we use those two last because they lead to an interesting conjecture—and another problem.

First Solution: Let S_e sum an even number of terms whose last is $2n$.

Let S_o sum an odd number of terms whose last is $2n+1$. Therefore $S_e = n(2n+1)$ and $S_o = (n+1)(2n+1)$.

Since both factors in both S_e and S_o are relatively prime, it follows that, if both sums are required to be squares, then both factors in both numbers must be squares.

Let n in S_e be q_2^2, and $(n+1)$ in S_o be q_1^2. Therefore, $S_o = q_1^2(2q_1^2-1)$, with $2q_1^2-1 = u_1^2$; and $S_e = q_2^2(2q_2^2+1)$, with $2q_2^2+1 = u_2^2$. This is the essence of the problem. We are required to find such numbers q_i^2 such that $2q_i^2 \pm 1 = u_i^2$.

Let us assume that there is a number $S_0 = q_0^2 u_0^2$, such that $2q_0^2+1 = u_0^2$.

We now show that if this is so, then there is an $S_1^2 = (q_0+u_0)^2(2q_0+u_0)^2$, such that if $(q_0+u_0)^2 = q_1^2$, then $(2q_0+u_0)^2 = 2q_1^2-1 = u_1^2$.

Proof. $(2q_0+u_0)^2 = 4q_0^2+4q_0u_0+u^2 = (2q_0^2+4q_0u_0+2u_0^2)+(2q_0^2-u_0^2)$. Since by hypothesis $2q_0^2-u_0^2 = (-1)$, therefore

$$(2q_0 + u_0)^2 = 2(q_0 + u_0)^2 - 1 = 2q_1^2 - 1 = u_1^2.$$

Obviously, this same process can be repeated on any $S_i^2 = q_i^2 u_i^2$ (with $+1$ and -1 alternating in the formula) so that if we can find some specific $S^2 = q_0^2 u_0^2$, we can provide an unending series of S_i^2.

We have, of course,

$$S_0^2 = 0^2(2 \cdot 0^2 + 1)^2, \quad \text{where} \quad qu = 0 \cdot 1, \text{ the sum of } 0 = 0$$

$$S_1^2 = (0 + 1)^2(2 \cdot 0 + 1)^2 = qu = 1 \cdot 1, \text{ the sum of } 1 = 1$$

$$S_2^2 = (1 + 1)^2(2 \cdot 1 + 1)^2 = qu = 2 \cdot 3, \text{ the sum of } 1 + 2 \cdots + 8 (\text{i.e. } 2 \cdot 2^2)$$

$$S_3^2 = 5^2 7^2$$

$$S_4^2 = 12^2 17^2$$

$$S_5^2 = (29 \cdot 41)^2$$

$$S_6^2 = (70 \cdot 99)^2$$

Second Solution:

(a) Given $S_o^2 = q^2(2q^2-1)$, with $2q^2-1 = u^2$.

Through the identity $2(a^2+b^2) = (a+b)^2+(b-a)^2$, and setting $q^2 = a^2+b^2$, we obtain $2q^2=(a+b)^2+(b-a)^2$.

Now, by letting $a+b=u$ and $b-a=1$, we obtain an immediate solution in which $S_o^2=(a^2+b^2)(a+b)^2$.

That is, there are solutions to S_o^2 in all cases where Pythagorean triangles have two legs, 'a' and 'b', that are consecutive integers.

(b) For $S_o^2=q^2(2q^2+1)$, we have

(1) $2q^2+1=u^2$, from which we derive

(2) $2q^2=(u+1)(u-1)$, and

$$2(q^2+1) = u^2+1$$

$$= 2\left[\left(\frac{u+1}{2}\right)^2+\left(\frac{u-1}{2}\right)^2\right] \quad \text{let} \quad \frac{u+1}{2}=a, \qquad \frac{u-1}{2}=b$$

then $q^2=2ab$. Setting this as the even leg of a rational right triangle, requires that the second leg be $a^2-b^2=u$, and the Pythagorean equation is

$$u^2+(q^2)^2=\left(\frac{u^2+1}{2}\right)^2=(q^2+1)^2$$

Thus we have demonstrated solutions for every rational triangle whose hypotenuse and even leg are consecutive integers, providing the even leg is a square number.

Third Solution. Returning to the equation $u^2-2q^2=\pm 1$, we recognize it as a particular case of Pell's equation which can be solved by expressing $(2)^{1/2}$ as a continued fraction, thus:

$$(2)^{1/2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}\cdots}}$$

where any convergent u/q offers a solution to the problem. The first convergents are

$$\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70}.$$

Pell's equation also solves the problem of finding integral right triangles whose legs are consecutive integers, thusly:

$$[(u+q)^2-q^2]^2+[2q(u+q)]^2=[(u+q)^2+q^2]^2.$$

Notes: One can also show that for every $S^2=(qu)^2$, the rational triangle $(q^2-u^2)^2+(2qu)^2=(q^2+u^2)^2$, is related to other $S_i^2=q_i^2u_i^2$.

After this demonstrated connection to $a^2+b^2=c^2$, what more natural question to ask than can $S=a^2+b^2$?

We obtain an immediate solution by letting $j=v^2$, that is, summing S to any square number.

For $v=2q$, we have $S=2q^2(4q^2+1)=(q^2+q^2)(4q^2+1)$.

Applying the identity, $(k^2+l^2)(m^2+n^2)=(km \pm ln)^2+(kn \mp lm)^2$, to S, we obtain S $(2q^2+q)^2+(2q^2-q)^2$. So that for every even square, $4q^2$, there is an $S=a^2+b^2$, with $a+b=4q^2$ and $a-b=2q$.

For $v=2q+1$, we have $2S=(2q+1)^2[(2q+1)^2+1]$.

But $(2q+1)^2+1=4q^2+4q+2=2\ [(q+1)^2+q^2]$, therefore,

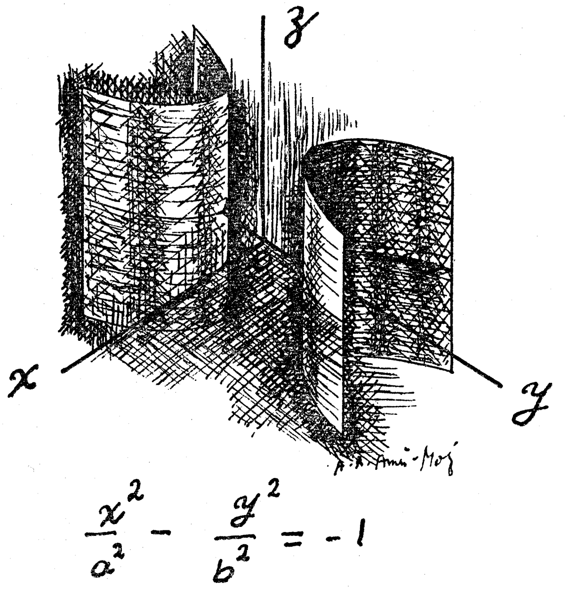
$$S = (2q + 1)^2[(q + 1)^2 + q^2],$$

which is, of course, the sum of two composite squares a^2+b^2, with $a+b=(2q+1)^2$ and $a-b=2q+1$.

The interesting conjecture mentioned above arises from the fact that the author has been unable to establish whether S can ever be an integral cube, or a fourth, fifth or nth-powered number. On the other hand, with $S_7=1^3+3^3$, $S_{13}=3^2+4^3$, and $S_{26}=2^3+7^3$, there is a hint—though admittedly very, very tenuous—that there may be an affirmative answer to the question of whether S can ever be the sum of two fourth, fifth or nth-powered numbers.

This is the basis for the following question, which may, perhaps, be looked upon as a "converse" to Fermat's Last Theorem stating that there exist no integral solutions to $x^n+y^n=z^n$ when $n \geqq 3$:

Are there integral solutions to $S=a^n+b^n$ for all $n \geqq 3$; and no solutions to $S=q^n$ for any $n \geqq 3$?



FIBONACCI FANCY

MAXEY BROOKE, Sweeny, Texas

Manipulation of the Fibonacci series produces an endless varity of interesting relationships. Here is one more.

$F_1 = 1 = (4 \times 0) + 1 = 4F_0 + 1$

$F_2 = 1 = (4 \times 0) + 1 = 4F_0 + 1$

$F_3 = 2 = (4 \times 0) + 2 = 4F_0 + 2$

****

$F_4 = 3 = (4 \times 0) + 3 = 4F_1 - 1$

$F_5 = 5 = (4 \times 1) + 1 = 4F_2 + 1$

$F_6 = 8 = (4 \times 2) = 4F_3$

$F_7 = 13 = (4 \times 5) + 1 = 4F_4 + 1$

$F_8 = 21 = (4 \times 5) + 1 = 4F_5 + 1$

$F_9 = 34 = (4 \times 8) + 2 = 4F_6 + 2$

****

$F_{10} = 55 = (4 \times 13) + 3 = 4F_7 + 3 = 4(F_7 + F_1) - 1$

$F_{11} = 89 = (4 \times 22) + 1 = 4(F_8 + 1) + 1 = 4(F_8 + F_2) + 1$

$F_{12} = 144 = (4 \times 36) = 4(F_9 + 2) = 4(F_9 + F_3)$

$F_{13} = 233 = (4 \times 58) + 1 = 4(F_{10} + 3) = 4(F_{10} + F_4) + 1$

$F_{14} = 377 = (4 \times 94) + 1 = 4(F_{11} + 5) + 1 = 4(F_{11} + F_5) + 1$

$F_{15} = 610 = (4 \times 152) + 2 = 4(F_{12} + 8) + 2 = 4(F_{12} + F_6) + 2$

******* ****** ******

$F_{16} = 987 = (4 \times 246) + 3 = 4(F_{13} + 13) + 3 = 4(F_{13} + F_7 + F_1) - 1$

$F_{17} = 1597 = (4 \times 399) + 1 = 4(F_{14} + 22) + 1 = 4(F_{14} + F_8 + F_2) + 1$

$F_{18} = 2584 = (4 \times 646) = 4(F_{15} + 36) = 4(F_{15} + F_9 + F_3)$

$F_{19} = 4181 = (4 \times 1045) + 1 = 4(F_{16} + 58) + 1 = 4(F_{16} + F_{10} + F_4) + 1$

$F_{20} = 6765 = (4 \times 1691) + 1 = 4(F_{17} + 93) + 1 = 4(F_{17} + F_{11} + F_5) + 1$

$F_{21} = 10946 = (4 \times 2736) + 2 = 4(F_{18} + 152) + 2 = 4(F_{18} + F_{12} + F_6) + 2$

*** *** **** *** ***

$F_{22} = 17711(4 \times 4427) + 3 = 4(F_{19} + 246) + 3 = 4(F_{19} + F_{13} + F_7 + F_1) - 1$

In general

$$F_n = 4(F_{n-3} + F_{n-9} + F_{n-15} + F_{n-21} \cdots) + A$$

If

$$n \equiv 0 \pmod 6,\ A = 0;\ n \equiv 1, 2, \text{ or } 5 \pmod 6,\ A = 1$$
$$n \equiv 3 \pmod 6,\ A = 2;\ n \equiv 4 \pmod 6,\ A = -1$$

CONSONANCE AND CONGRUENCE

CHARLES W. VALENTINE, Electric Boat Division, General Dynamics Corp.

This paper is an account of some of the mathematical properties of the equally tempered musical scale, properties which illustrate in an elementary way the use of Gaussian congruences and the closely related theory of finite cyclic groups.

The mathematics is in no sense profound, but the mathematician with musical interests or the musician with interests in mathematics may experience in reading it a little of that sort of pleasure associated with meeting an old friend in an unexpected place.

In order to avoid digressions into the theory of music, the discussion is framed completely in terms of musical notation. However, many of the properties discussed are not merely of notational interest but are of intrinsic significance in musical structure, particularly in modulation.

We first impose the following mathematical notation on the chromatic scale:

C	*C*♯	*D*	*D*♯	*E*	*F*	*F*♯	*G*	*G*♯	*A*	*A*♯	*B*	*C*
	(*D*♭)		(*E*♭)			(*G*♭)		(*A*♭)		(*B*♭)		
$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$	$\{8\}$	$\{9\}$	$\{10\}$	$\{11\}$	$\{0\}$

Here $\{k\}$ denotes the congruence class of k modulo 12.

To fix the ideas, let us consider the scale of *C* major.

C	*D*	*E*	*F*	*G*	*A*	*B*	*C*
$\{0\}$	$\{2\}$	$\{4\}$	$\{5\}$	$\{7\}$	$\{9\}$	$\{11\}$	$\{0\}$

The intervals between successive notes (congruence classes) follow the pattern 2 2 1 2 2 2 1 which is called the major diatonic scale. The pattern is extended below.

C	*D*	*E*	*F*	*G*	*A*	*B*	*C*	*D*	*E*	*F*	*G*
$\{0\}$	$\{2\}$	$\{4\}$	$\{5\}$	$\{7\}$	$\{9\}$	$\{11\}$	$\{0\}$	$\{2\}$	$\{4\}$	$\{5\}$	$\{7\}$

1 * 2 2 1 2 2 2 1 * 2 2 1 2 2

The note above the asterisks is called the tonic note or key and occupies a unique niche in the pattern of intervals; that is, due to the asymmetry of the pattern within each grouping of seven intervals, it is impossible to confuse any other note with the tonic note.

Changes of key are effected by interchanging adjacent intervals in one of two ways. The first is called sharping or adding a sharp and is illustrated below.

· · · 1 * 2 2 $\overline{1\quad 2}$ 2 2 1 * 2 2 $\overline{1\quad 2}$ 2 · · ·

· · · 1 2 2 $\overline{2\quad 1}$ * 2 2 1 2 2 $\overline{2\quad 1}$ * 2 · · ·

Remembering that the unit interval is called a half-tone, it is evident that adding a sharp to a given key shifts the key or tonic note upwards seven half-tones (a perfect fifth) or, equivalently, downward five half-tones (a perfect fourth).

The second way is called flatting or adding a flat and is illustrated below.

$$\cdots 1 \; * \; 2 \quad 2 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \; * \; 2 \quad 2 \quad 1 \quad 2 \quad 2 \cdots$$

$$\cdots 2 \quad 2 \quad 2 \quad 1 \; * \; 2 \quad 2 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \; * \; 2 \quad 2 \cdots$$

Flatting shifts the tonic note upward (downward) by five (seven) half-tones.

Starting from any given key it is possible to add either sharps or flats indefinitely. This is only of theoretical interest, however, since there are actually only twelve keys, and a musician, like anyone else, is interested in doing things in the simplest manner. An illustration of the commonly used notations for the twelve keys is:

C	*G*	*D*	*A*	*E*	*B*	*F*♯	*C*♯					
	1♯	2♯	3♯	4♯	5♯	6♯	7♯					
{0}	{7}	{2}	{9}	{4}	{11}	{6}	{1}	{8}	{3}	{10}	{5}	{0}
					7♭	6♭	5♭	4♭	3♭	2♭	1♭	
					C♭	*G*♭	*D*♭	*A*♭	*E*♭	*B*♭	*F*	

Notice that for each of the keys where both flat and sharp notations are given, the number of sharps plus the number of flats equals twelve. This is a useful rule known empirically to many musicians.

The validity of this rule is dependent on the truth of the congruence relation:

$$7k \equiv 5(12n - k) \pmod{12}$$

which follows from the addition of the obvious relations:

$$12k \equiv 5(12n) \pmod{12}$$

$$-5k \equiv -5k \pmod{12}.$$

A more satisfying explanation of this rule is obtained by recognizing that sharping and flatting may be considered inverse elements of a finite cyclic group of order 12. Now if p is any member of a cyclic group of order 12 then so is p^{-1}. Hence if u is the identity of the group,

$$p^{-12} = u.$$

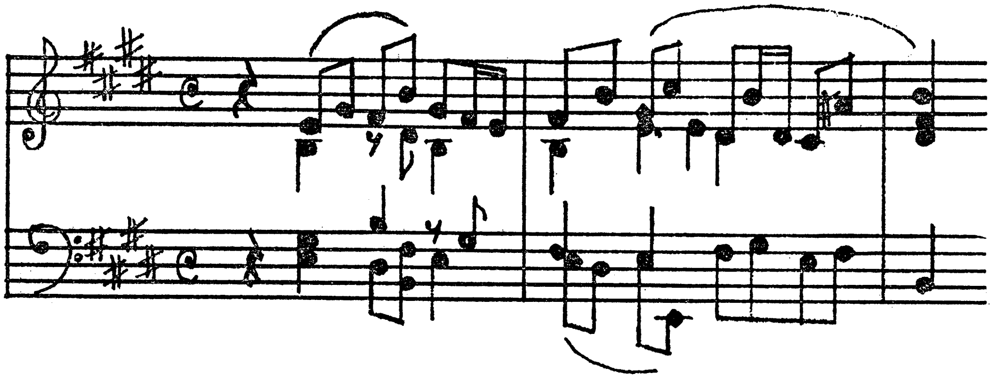
Hence

$$p^k \cdot p^{-12} = p^k \cdot u$$

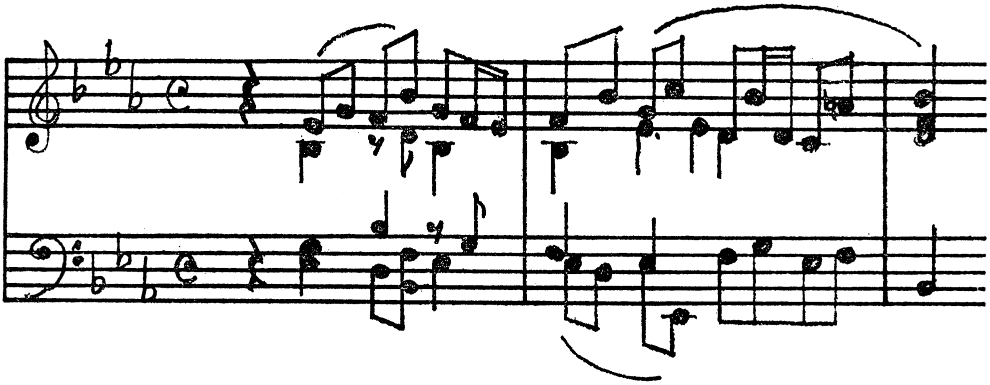
or

$$p^{-(12-k)} = p^k.$$

Consider another application of congruences. A pianist may fail to note the key signature of a passage such as the following:



and play the passage as if it were written:



The result is that the music is played a half-tone lower than was intended by the composer; to use our mathematical notation, the piece is played in the key of $\{n-1\}$ instead of the key of $\{n\}$.

The number of sharps plus the number of flats in the example above equals seven. This suggests the theorem:

$$[7k \equiv n \pmod{12}] \Rightarrow [5(7-k) \equiv n-1 \pmod{12}].$$

Now

$$7k \equiv -5k \pmod{12};$$

hence

$$-5k \equiv n \pmod{12},$$

and

$$36 - 5k \equiv n \pmod{12};$$

or

$$35 - 5k \equiv n - 1.$$

Factoring we have

$$5(7 - k) \equiv n - 1 \pmod{12},$$

as was to be proven.

Returning to the relation

$$7k \equiv 5(12n - k) \pmod{12},$$

we notice that for any p, q such that $p+q=12$,

$$pk \equiv q(12n - k) \pmod{12}.$$

However, we also notice that, using the customary notation for the greatest common divisor, $(7, 12)=(5, 12)=1$. Is there any reason to expect that the numbers associated with the flatting and sharping operators should be relatively prime to twelve? Yes; given a method of proceeding from key to key, it is desirable that all of the twelve keys may be reached by continued application of this method. Now if p were the shift in key associated with a given method and for example

$$(p, 12) = 2,$$

then we would have

$$6p \equiv 0 \pmod{12}.$$

We would return to our original key after only six applications of the method, and there would be six keys which would never be reached.

The reason we seek is therefore implicit in the following theorem from number theory

$$(p, q) \cdot [p, q] = pq$$

where $[p, q]$ denotes the least common multiple of p and q.

Moreover, the reason is explicitly stated in the following theorem from group theory: *If n is the group of integers (mod n) and $k \in n$, then the subgroup of n generated by k is of order $n/(k, n)$.*

Now $(11, 12)=(1, 12)=1$. Solving the congruences:

$$7x \equiv 1 \pmod{12}: \qquad x = 7$$
$$5x \equiv 1 \pmod{12}: \qquad x = 5$$
$$5x \equiv 11 \pmod{12}: \qquad x = 7$$
$$7x \equiv 11 \pmod{12}: \qquad x = 5,$$

we see that if we introduced flats or sharps five or seven at a time instead of one at a time, then it would still be possible to reach all keys starting from any given one of them. There is another reason, however, why flats and sharps are added one at a time; adding a sharp or flat to a given key translates the tonic note, and hence the entire scale, but actually changes only two notes. Consequently the new key and the original key have five notes in common. Thus adding one sharp or flat transforms a given diatonic scale to a new diatonic scale with a minimal change in the actual notes involved.

We notice from our results above that

$$7^2 \equiv 1 \pmod{12}$$
$$5^2 \equiv 1 \pmod{12}.$$

This suggests the theorem:

$$[(m, 12) = 1] \Rightarrow m^2 \equiv 1 \pmod{12},$$

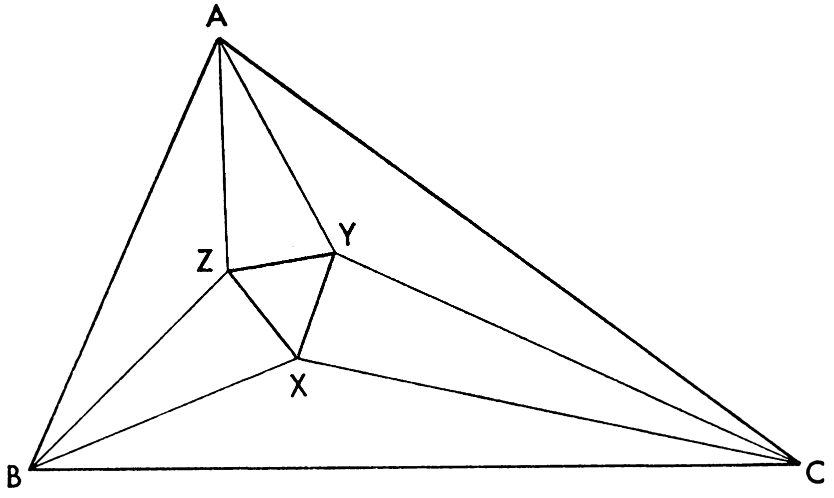
which in turn suggests the following slightly stronger theorem:

$$[(m, 6) = 1] \Rightarrow m^2 \equiv 1 \pmod{24}.$$

Both of these simple theorems may be proven by enumeration of cases, but more interesting proofs are possible in which no enumeration is necessary.

A SIMPLE PROOF OF THE MORLEY THEOREM

LEON BANKOFF, Los Angeles, California



THEOREM. The intersections of the adjacent internal angle trisectors of a triangle are the vertices of an equilateral triangle.

Proof. We make use of the following relations:

$$(1)\quad \sin AYC = \sin\left(\pi - \frac{A+C}{3}\right) = \sin\left(\frac{A+C}{3}\right) = \sin\left(\frac{\pi - B}{3}\right) = \sin\left(\frac{2\pi + B}{3}\right)$$

$$(2)\qquad \sin 3\theta = \sin\theta\cos 2\theta + \sin 2\theta\cos\theta = \sin\theta(3\cos^2\theta - \sin^2\theta)$$

$$= 4\sin\theta\left(\frac{\sqrt{3}\cos\theta + \sin\theta}{2}\right)\left(\frac{\sqrt{3}\cos\theta - \sin\theta}{2}\right)$$

$$= 4\sin\theta\sin\left(\frac{\pi}{3} + \theta\right)\sin\left(\frac{\pi}{3} - \theta\right).$$

Applying the Sine Law in triangle AYC, we have

$$AY\sin\left(\frac{\pi - B}{3}\right) = AC\sin\frac{C}{3} = 2R\sin B\sin\frac{C}{3},$$

where R is the circumradius of triangle ABC, and by eq. (2),

$$AY = 8R\sin\frac{B}{3}\sin\frac{C}{3}\sin\left(\frac{\pi + B}{3}\right).$$

Similarly,

$$AZ = 8R\sin\frac{B}{3}\sin\frac{C}{3}\sin\left(\frac{\pi + C}{3}\right),$$

whereupon

$$AZ/AY = \sin\left(\frac{\pi + C}{3}\right)\Big/\sin\left(\frac{\pi + B}{3}\right).$$

But

$$\measuredangle AZY + \measuredangle AYZ = \pi - \frac{A}{3} = \frac{2\pi + B + C}{3} = \frac{\pi + C}{3} + \frac{\pi + B}{3}.$$

Hence

$$\measuredangle AYZ = \frac{\pi + C}{3} \text{ and } \measuredangle AZY = \frac{\pi + B}{3}.$$

Similarly,

$$\measuredangle BZX = \frac{\pi + A}{3};\quad \measuredangle BXZ = \frac{\pi + C}{3};\quad \measuredangle CYX = \frac{\pi + A}{3};\quad \measuredangle CXY = \frac{\pi + B}{3}.$$

The sum of the known angles around points X, Y, Z is in each case 300°, so that each angle of triangle XYZ is 60°.

ON CHROMATIC BIPARTITE GRAPHS

J. W. MOON AND L. MOSER, University of Alberta

Suppose there are given two distinct sets of n points each, such that joining each pair of points, not both in the same set, is a line which is colored either red or blue. In such a configuration, called a chromatic bipartite graph, what is a lower bound for the total number of quadrilaterals whose four sides are all of the same color?

In ordinary chromatic graphs the analogous problem with respect to triangles was solved by Goodman [1], and later by Sauvé [2]. In that case the number of monochromatic triangles may be expressed as a function of, say, the number of red lines incident on each point, but it is easy to construct two chromatic bipartite graphs, each having the same number of red lines incident on each point, which do not contain the same number of monochromatic quadrilaterals.

For convenience we may equivalently interpret our problem in terms of $n\times n$ matrices all of whose elements are either 0 or 1 and where $P(n)$ and $Q(n)$ denote the number of 2×2 minors all of whose elements are 1's or 0's respectively. We will prove the following:

Theorem

$$(1)\qquad P(n)+Q(n)\geqq\begin{cases}2u^2(u-1)(4u-3), & \text{if } n=4u;\\ 2u^3(4u-3), & \text{if } n=4u+1;\\ u(2u+1)(4u^2-u-1), & \text{if } n=4u+2;\\ u^2(2u+1)(4u+3), & \text{if } n=4u+3;\end{cases}$$

where u is a nonnegative integer.

Proof: In any $n\times n$ matrix of 0's and 1's, $A=\|a_{ij}\|$, let r_i denote the ith row sum. Then the number of pairs of columns, k and l, for which there exists a row i such that $a_{ik}=a_{il}$ is

$$(2)\qquad \sum_{i=1}^{n}\left[\binom{r_i}{2}+\binom{n-r_i}{2}\right],\qquad \text{counting multiplicities.}$$

If the $\binom{n}{2}$ pairs of columns have been numbered, let t_v and h_v denote the number of times the vth pair has been counted in (2), with $a_{ik}=a_{il}=1$ and $a_{ik}=a_{il}=0$, respectively.

Then

$$(3)\qquad \sum_{v=1}^{\binom{n}{2}} t_v=\sum_{i=1}^{n}\binom{r_i}{2},\qquad \sum_{v=1}^{\binom{n}{2}} h_v=\sum_{i=1}^{n}\binom{n-r_i}{2},$$

and

$$(4)\qquad P(n)+Q(n)=\sum_{v=1}^{\binom{n}{2}}\left[\binom{t_v}{2}+\binom{h_v}{2}\right].$$

Hence, to minimize $P(n)+Q(n)$, we should have $\sum_{i=1}^{n} r_i$ as nearly equal to $\sum_{i=1}^{n}(n-r_i)$ as possible, the r_i's as nearly equal to each other as possible, the t_v's as nearly equal to each other as possible and similarly for the h_v's.

When $n=4u$ the smallest possible value for $P(n)+Q(n)$ would occur when all $r_i=2u$ and t_v was equal to u for $2u(3u-1)$ values of v and $u-1$ for the remaining $2u^2$ values of v, since

$$u[2u(3u-1)]+(u-1)(2u^2)=4u\binom{2u}{2}=\sum_{i=1}^{n}\binom{r_i}{2},$$

and similarly for the h_v's. Hence, for any matrix in this case

$$(5)\qquad P(n)+Q(n)\geqq 2\left[2u(3u-1)\binom{u}{2}+2u^2\binom{u-1}{2}\right]=2u^2(u-1)(4u-3).$$

In the same way it is found that when $n=4u+2$ the minimum would be when all $r_i=2u+1$, t_v equalled u for $(2u+1)(3u+1)$ values of v and $u+1$ for the remaining $u(2u+1)$ values, and similarly for the h_v's. In this case

$$(6)\qquad P(n)+Q(n)\geqq 2\left[(2u+1)(3u+1)\binom{u}{2}+u(2u+1)\binom{u+1}{2}\right]$$
$$=u(2u+1)(4u^2-u-1).$$

When $n=4u+3$ it is obviously impossible to have

$$\sum_{i=1}^{n} r_i=\sum_{i=1}^{n}(n-r_i).$$

The nearest we could come to equality would be when, say, r_i was $2u+2$ for $2u+2$ values of i, and $2u+1$ for the remaining $2u+1$ values. The other conditions in this case would require that t_v equal $u+1$ for $2(u+1)(2u+1)$ values of v and u for the remaining $(2u+1)^2$ values, and that h_v equal $u+1$ for $(2u+1)^2$ values of v and u for the others. Hence,

$$(7)\qquad P(n)+Q(n)\geqq(2u+1)(4u+3)\left[\binom{u+1}{2}+\binom{u}{2}\right]=u^2(2u+1)(4u+3).$$

When $n=4u+1$ the same sort of reasoning implies that the minimum value of $P(n)+Q(n)$ would occur when, say, r_i was equal to $2u+1$ for $2u+1$ values of i and $2u$ for the remaining values, t_v was equal to u for $8u^2+u$ values of v and $u+1$ for the u other values, and h_v was equal to u for $8u^2+u$ values of v and $u-1$ for the remaining u values. As a lower bound this gives

$$(8)\qquad P(n)+Q(n)\geqq 2(8u^2+u)\binom{u}{2}+u\left[\binom{u+1}{2}+\binom{u-1}{2}\right]$$
$$=u[(4u+1)(u-2)(2u)+1].$$

But it is impossible for equality ever to be attained, if $u>0$. To see this we notice that, by symmetry, a matrix for which $P(n)+Q(n)$ is to attain the minimum in (8) must have, in this case, $2u$ columns whose sum is $2u$. For each of the $\binom{2u}{2}$ pairs of such columns t_v is u or $u+1$ and h_v is $u-1$ or u. Upon attempting to construct two columns, in an $4u+1\times 4u+1$ matrix, each containing $2u$ 1's and $2u+1$ 0's it is easily seen that none of these combinations of t_v and h_v are possible and that the next best combinations, from the point of view of minimizing the sum in (4), are $t_v=u$, $h_v=u+1$ and $t_v=u-1$, $h_v=u$. For each $h_v=u+1$, for some of these pairs, there must be an extra $h_v=u-1$ in order to preserve equality in (3) and a similar remark can be made with respect to those t_v which equal $u-1$. Clearly there were enough t_v's and h_v's set equal to u originally to permit all the necessary changes. Since

$$\binom{u+1}{2}+\binom{u-1}{2}=2\binom{u}{2}+1,$$

we see that whichever change we make, we are forced to increase the sum in (4) by 1 for each of the $\binom{2u}{2}$ pairs of columns whose column sum is $2u$.

Adding this to the bound obtained previously we have, when $n=4u+1$,

$$P(n)+Q(n)\geqq 2u^3(4u-3). \tag{9}$$

It would be of interest to be able to give a general construction which would show that the lower bounds in Theorem 1 were sharp, or to give larger bounds if they are not. No such simple consideration as applied to the case $n=4u+1$ suffices to show that the bounds in (1) are not attainable. By examples it can be shown that the bounds are sharp for $n\leqq 9$.

The original problem, as well as the argument, admits of obvious generalizations.

References

1. A. W. Goodman, On sets of acquaintances and strangers at any party, Amer. Math. Monthly, **66** (1959) 778–783.

2. Léopold Sauvé, On chromatic graphs, Amer. Math. Monthly, **68** (1961) 107–111.

CONCERNING A CONJECTURE OF R. CROCKER

ANDRZEJ MĄKOWSKI, Warsaw, Poland

The special case of Crocker's conjecture stated in [1] is that every sufficiently large integer is a sum of a prime and at most two squares. It is a conjecture of Hardy & Littlewood which has been proved by Yu. V. Linnik (for reference see the *Mathematical Reviews*, **22** (1961) #10963-4).

Reference

1. Roger Crocker, A theorem concerning prime numbers, MATHEMATICS MAGAZINE **34** (1961 316, 344.

THE STORY OF $(BGG)_i$ $(i=1, 2, 3)$

LELAND H. WILLIAMS, Florida State University

(This mathematical revision of a well-known classic was inspired by R. F. Rinehart's contribution [1] to the field of mathematical revisions of well-known classics. This revision is not as mathematically elegant as his, but the abbreviation achieved is more striking in that the story is almost entirely iterative.)

$\exists\ t_0 \ni$ at $t=t_0\ \exists$ on the rocky side of a stream $(BGG)_i$ $(i=1, 2, 3)$, where $\{(BGG)_{i+1} > (BGG)_i, i=1, 2\}$. On the other side of the stream there was an abundance of cool, green grass. A mean old troll guarded the bridge leading from one side to the other. Until $t=t_0$, he had eaten all who attempted to cross the bridge.

$\{\ \exists\ t_i(t_i > t_{i-1}) \ni$ at $t=t_i$, $(BGG)_i$ decided to risk being eaten and cross the bridge anyway. The troll asked who was crossing the bridge. The answer was $(BGG)_i$. The troll threatened to eat $(BGG)_i$, who $[i \leqq 2 \to *,\ i=3 \to **]$.

$*$ responded with the statement that the much fatter and tastier $(BGG)_{i+1}$ would be along soon. The troll then allowed $(BGG)_i$ to pass over the bridge.$\}$, $(i=1, 2, 3)$.

$**$ dared him to try. The ensuing battle ended at $t=t_4$ $(t_4 > t_3)$ with $(BGG)_3$ butting the troll off the bridge into the stream.

The troll was never seen again and $\forall\ t \ni t > t_4$ the $(BGG)_i$, $(i=1, 2, 3)$ were able to eat grass whenever they liked.

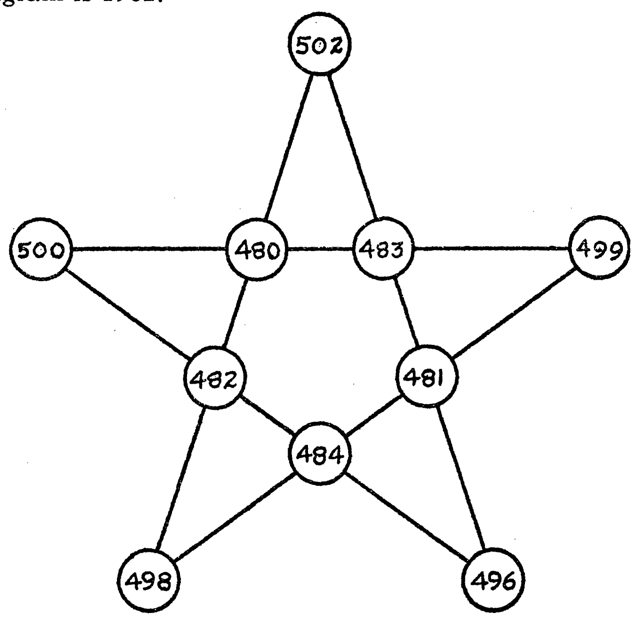
Reference

1. R. F. Rinehart, The modern mathematical approach to LR²H, MATHEMATICS MAGAZINE, 32 (1959) 151–152.

A MAGIC PENTAGRAM FOR 1962

CHARLES W. TRIGG, Los Angeles City College

The sum of the integers at the intersections along each of the five lines forming the pentagram is 1962.



EDITED BY ROTHWELL STEPHENS, Knox College

This department is devoted to the teaching of mathematics. Thus, articles of methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as a teacher, are interested, or questions which you would like others to discuss, should be sent to Rothwell Stephens, Mathematics Department, Knox College, Galesburg, Illinois.

A SIMPLE PROOF OF THE FORMULA FOR SIN $(A+B)$

NORMAN SCHAUMBERGER, Bronx Community College of the City University of New York

The following proof was communicated by Professor Jesse Douglas of the City College of the City University of New York, who has been using it for many years in his classes. It is composed of the following four ingredients, each a standard theorem of trigonometry:

(1) The projection law for triangles (each side equals the algebraic sum of the projections upon it of the other two): $c=a\cos B+b\cos A$;

(2) The sine law:

$$\frac{\sin C}{c}=\frac{\sin A}{a}=\frac{\sin B}{b};$$

(3) $A+B+C=\pi$;

(4) $\sin(\pi-\theta)=\sin\theta$.

Multiplying the terms in (1) by the corresponding ones in (2), we get: $\sin C=\sin A\cos B+\cos A\sin B$. By (3) and (4), $\sin C=\sin(A+B)$; hence

$$\sin(A+B)=\sin A\cos B+\cos A\sin B. \tag{*}$$

The addition theorem is thus established for all A, B that may occur as two angles of a triangle, i.e., $A+B<\pi$, $A>0$, $B>0$. The case A or $B=0$ and the case $A+B=\pi$ are immediate by the formulas $\sin 0=0$, $\cos 0=1$, $\cos(\pi-\theta)=-\cos\theta$, and (4). The extension to the case $0<A<\pi$, $0<B<\pi$, $A+B>\pi$ is established by applying (*) to $\pi-A$, $\pi-B$, since $(\pi-A)+(\pi-B)=2\pi-(A+B)<\pi$. (*) is now proved for $0\leqq A<\pi$, $0\leqq B<\pi$.

We may then extend (*) to all A, B such that $0\leqq A<2\pi$, $0\leqq B<2\pi$, by applying (*), as so far proved, to $A-\pi$, B; A, $B-\pi$; $A-\pi$, $B-\pi$; according as A, or B, or both, are $\geqq\pi$. One uses $\sin(\theta-\pi)=-\sin\theta$, $\cos(\theta-\pi)=-\cos\theta$, $\sin(\theta-2\pi)=\sin\theta$.

(*) now follows universally by periodicity, since for arbitrary A, B and appropriate integers m, n, the angles $A+2m\pi$, $B+2n\pi$ belong to the interval $0\leqq\theta<2\pi$.

IMPLICATIONS OF THE NEW HIGH SCHOOL MATHEMATICS PROGRAMS FOR TEACHER EDUCATION

LAWRENCE A. RINGENBERG,* Eastern Illinois University

1. Proper Placement. Prospective high school mathematics teachers need a background of algebra, geometry, trigonometry, analytic geometry, and calcu-

* Presented at Champaign, Illinois, October 13, 1961, at a meeting of the Illinois Council of Teachers of Mathematics.

lus. A good program of four years of high school mathematics provides the necessary background for a capable college freshman to enroll in courses in analytics and calculus. Some students, however, do not have this background when they enter college, perhaps due to inadequacies in their high school program, or perhaps due to a lack of motivation or ability on the part of the student. The implication, as I see it, is that placement tests are necessary if we are to enroll freshmen in the proper mathematics courses. For those who are not prepared to enter analytics and calculus we should be merciful to the maximum extent of about five semester hours work in college algebra and trigonometry. Students who are not prepared to enter courses in college algebra and trigonometry should be given an opportunity to self-educate themselves without credit. Students who have had work in analytics and calculus in high school should be tested for proper placement, or placed according to CEEB test scores. A year's work in the calculus must be included in the mathematics major as well as the mathematics minor, since a background in the concepts of the calculus is necessary for so many of the advanced courses which the student should take in his advanced undergraduate years and/or in his graduate program.

2. Algebra. The modern courses in high school algebra are characterized by a new vocabulary, an increased emphasis on algebraic structure, and more attention to deductive reasoning and basic concepts. College mathematics courses should provide the prospective teacher with a background so that he can assume with confidence an assignment to teach a good high school algebra course, particularly one of these courses with a new look. Perhaps it can be done within the framework of old college courses with modern texts; perhaps it requires new courses. At any rate I think it is necessary that the college teacher of the prospective high school teacher be familiar with the content of these new high school courses. At the University of Illinois prospective high school teachers take one or more courses which include units in which UICSM materials are used. At Eastern Illinois University prospective teachers are enrolled in Mathematics 125 during one of the quarters in the freshman year. This course is an Introduction to Mathematics. Topics which have been included in this course are the following ones: various numeral systems, numerals to various bases, the mathematics of computation in the natural number system (developed from a set of postulates), the number systems of elementary mathematics (natural, integral, rational, real, complex) developed constructively starting from the natural numbers, the properties of a mathematical system, the fundamental rules of inference, and an introduction to the concepts of group and field. The concepts of a set, the basic operation on sets, the set builder notation, and the fundamentals of the propositional calculus are introduced in the course. Also included are the concepts of a function, the solution set of an equation, and relation. In this connection some work on graphing inequalities with application to linear programming is included. All these things will be a part of the background of most entering freshmen in the near future. In the meantime we must include this work in the college program. Including it in the freshman year concurrently with a course in trigonometry, or analytics, or calculus, provides the student with a degree of mathematical maturity which will benefit him in all of his mathematical courses. It will help him particularly when he takes advanced undergraduate courses in abstract modern algebra.

The mathematics professor who is teaching a prospective teacher can help him to grow in stature as regards his competence in modern elementary algebra; he can give him this help regardless of what the course is. He can be careful that his students are conscious of the domains of functions which are considered. He can encourage the use of the set builder notation in a great variety of situations. He can point out instances of theorems and their opposites, converses, and contrapositives. An example from Sherwood and Taylor's *Calculus* comes to mind in this connection:

Theorem. *A necessary condition that a series of real numbers be convergent is that the limit of the nth term of the series as n approaches infinity be zero; in other words, if the limit of the nth term of a series of real numbers, as n approaches infinity, is not zero, then the series is divergent.*

The student who has seen the logical equivalence of a theorem and its contrapositive developed using truth tables will understand this concept and the calculus theorem both just a little bit better if he is made aware of the fact that Sherwood and Taylor's "other words" are but a statement of the contrapositive. The mathematics professor should be careful in regard to impressions he might leave regarding deductive reasoning. It is good for the student to see the mind of the professor at work. But sometimes there is a temptation to not clean up the mess once the desired result is obtained. Instead of throwing out the negative root at the end because it "clearly makes no sense," it would be better to return to the beginning of the solution and add another sentence (equation or inequality) which restricts the domain of x. Then the solution set of the set of sentences is the desired result. And there is no confusion in regard to the logical basis for throwing out meaningless answers.

The background of the prospective mathematics teacher should include at least six semester hours in advanced undergraduate courses in abstract algebra. These courses should include work in algebraic structure, groups, rings, fields, ordered fields, matrices, and vector spaces. Recommended guides for these courses have been provided by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America.

3. Algebra in the Eighth Grade. Many large schools start an advanced placement or accelerated program in mathematics in the seventh or eighth grade. In a typical program one or two sections of gifted eighth graders are placed in a course in ninth grade algebra. This trend will increase rapidly. An implication of this is that at least one of the eighth grade teachers should have a background which includes a strong minor in mathematics, including the work discussed above, to familiarize the prospective teacher with the new vocabulary and the content of high school algebra and at least one course in abstract modern algebra.

4. Geometry. The modern high school geometry course for tenth or eleventh graders presupposes a background of work in informal geometry. Geometric concepts should be treated early in the grades along with work in arithmetic. A considerable portion of the seventh and eighth grade mathematics program should be devoted to informal geometry. Too frequently the units on geometry at the end of the text are omitted. All of us should think in terms of grade school mathematics instead of grade school arithmetic. One implication of this is that the curriculum for the preparation of elementary teachers should include a course, or a substantial unit of work, in geometry with particular emphasis on the historical development of geometry and the foundations of geometry. The elementary school teacher should understand clearly that "truth" in geometry can be revealed in various ways. Informal geometry has its basis in observation, intuition, analogy, experiment, and inductive reasoning. The methods of informal geometry are the methods of science. The geometry course for the prospective elementary school teacher should include a complete development of the area and volume formulas for elementary figures and of ruler and compass constructions. Since the new high school geometry course is quite different from the traditional course, the prospective elementary teacher would profit greatly from a course which includes a unit on the content of the new geometry courses. The grade school teacher ought to be familiar with the course his pupils will take as tenth graders.

What about the preparation in geometry for the prospective senior high school mathematics teacher. Traditionally, the curriculum for prospective mathematics teachers has included a course in college geometry. This course consists largely of topics in advanced and modern Euclidean geometry. This is a good course but it is not enough. The new high school geometry courses are so different from the traditional courses that the college curriculum must include a course or a substantial unit on the foundations of geometry and the methods and content of high school geometry. The modern high school geometry course is characterized by a higher level of mathematical rigor than the traditional course. The prospective teacher must feel at home in a course in which there are no tacit assumptions, a course in which a consideration of existence of points and lines, of betweenness relations, of separation properties is included, a course which recognizes Euclidean formal geometry as one of several geometries which are equally sound mathematically, a course which includes an introduction to coordinates and vector methods. Until such time as most high school youth take one of these new high school geometry courses, a course to familiarize the prospective mathematics teacher with the new courses should be included in the college program. This might be accomplished in a course on the content and methods of school geometry, or it might be accomplished in a course on the foundations of mathematics.

The new SMSG experimental course in geometry includes a development which concludes with the following theorem.

THEOREM. *Suppose a one dimensional coordinate system is fixed on a line l. If*

A, B are two points on l with coordinates a and b, then $l = \{P_x$: k is a real number and $x = a + k(b-a)\}$.

Proof. Suppose P is in the ray $\overrightarrow{AB}$. If $P=A$ then $x=a$ and $k=0$. If $P=B$ then $x=b$ and $k=1$. If $P \neq A$, then $AP/AB=k$ for some positive number k. Suppose $x>a$. Then $b>a$ and $AP=x-a$, $AB=b-a$, $x-a=k(b-a)$, $x=a+k(b-a)$. If on the other hand, $x<a$, then $b<a$, and $AP=a-x$, $AB=a-b$, $a-x=k(a-b)$ and $x=a+k(b-a)$. Suppose next that P is not in the ray $\overrightarrow{AB}$, but in the opposite ray. Again $AP/AB=k$ for some positive number k. Suppose $x>a$. Then $b<a$ and $AP=x-a$, $AB=a-b$, $x-a=k(a-b)$, $x=a-k(b-a)$. If $x<a$, then $b>a$, $AP=a-x$, $AB=b-a$, $a-x=k(b-a)$, $x=a-k(b-a)$. It follows that

$$\overrightarrow{AB} = \{P_x\colon x = a + k(b-a),\ k \geqq 0\}$$

$$\text{Opp. } \overrightarrow{AB} = \{P_x\colon x = a - k(b-a),\ k \geqq 0\} = \{P_x\colon x = a + k(b-a),\ k \leqq 0\}$$

$$\overleftrightarrow{AB} = \{P_x\colon x = a + k(b-a),\ k \text{ is real}\}.$$

This theorem forms a cornerstone of the later work in coordinate geometry for two and three dimensions. Of course the development leading to this theorem is careful and deliberate. And the Teachers Commentary provides further assistance for the teacher. But the teacher who has been forewarned in college to expect new things in geometry, and who has seen some of these new things, will accept an assignment to teach one of the new geometry courses with more confidence than if he had no preparation for it.

5. Analytic Geometry and Calculus. Many of the large high schools today participate in the advanced placement program in mathematics, and more of them will in the future. This implies that at least one teacher in the high school should be qualified to teach a college level calculus course. The teacher of calculus should have a background of college work in physics and preferably also one or more other areas, such as engineering, chemistry, and economics, in which there are interesting and useful applications of the calculus. The teacher of calculus should have a depth of mathematical knowledge which will make him an inspiring teacher, one who has confidence in himself as regards competence in mathematics, and one who should be able to read the text book critically. He should be aware of the fact that the author has had to make some compromises, that he hasn't told the whole story. He should know what a continuous function (a single-valued real function of a real variable) is. He should be able to prove that any polynomial in x defines a continuous function of x. He should know about open and closed sets of points, transfinite cardinal numbers, uniform continuity, the Heine-Borel covering theorem; he should be able to prove that the Riemann integral of a continuous function over a closed interval exists. He should know about generalizations of the concept of continuity, about Lebesgue measure, about the fundamental theorem of the calculus and its extensions or generalizations. These are specific examples listed to suggest that the calculus teacher, whether he is a high school teacher or a college teacher, needs a background in real variables. The high school which does not have a teacher with such a background should not participate in the advanced placement program in mathematics.

6. Probability and Statistics. Due largely to the work of the Commission on Mathematics of the College Entrance Examination Board a course in probability and statistics is now included in some high school curricula. Since this is such an important subject in many college curricula, the trend to teach some statistics in high school will increase. An obvious implication is that the curricula for preparation of high school mathematics teachers must include one or more courses in statistics. The senior high school teacher should have a background of at least six semester hours work in statistics, while the junior high school teacher should have a background of at least three semester hours in this area.

AN EXPANSION OF THIRD ORDER DETERMINANTS

CHARLES W. TRIGG, Los Angeles City College

A third order determinant,

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

may be expanded and evaluated as follows:

$$\begin{aligned}
\Delta &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \\
&= \big[(a_1b_1b_2c_3 - a_2b_1^2c_3 - a_1b_2b_3c_1 + a_2b_1b_3c_1) \\
&\qquad - (a_1b_1b_3c_2 - a_3b_1^2c_2 - a_1b_2b_3c_1 + a_3b_1b_2c_1)\big]/b_1 \\
&= \big[(a_1b_2 - a_2b_1)(b_1c_3 - b_3c_1) - (a_1b_3 - a_3b_1)(b_1c_2 - b_2c_1)\big]/b_1 \\
&= \left\{ \begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix} \cdot \begin{vmatrix} b_1c_1 \\ b_3c_3 \end{vmatrix} - \begin{vmatrix} a_1b_1 \\ a_3b_3 \end{vmatrix} \cdot \begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix} \right\} /b_1 \\
&= (C_3A_2 - C_2A_3)/b_1,
\end{aligned}$$

where the capital letters represent the minors of the corresponding lower case letter elements. In like manner, or by cyclic permutations of letters and subscripts, eight more similar expressions may be obtained. That is,

$$\begin{aligned}
\Delta &= (A_1C_3 - A_3C_1)/b_2 = (A_1C_2 - A_2C_1)/b_3 \\
&= (B_2C_3 - B_3C_2)/a_1 = (B_1C_3 - B_3C_1)/a_2 = (B_1C_2 - B_2C_1)/a_3 \\
&= (A_2B_3 - A_3B_2)/c_1 = (A_1B_3 - A_3B_1)/c_2 = (A_1B_2 - A_2B_1)/c_3
\end{aligned}$$

MISCELLANEOUS NOTES

EDITED BY ROY DUBISCH, University of Washington

Articles intended for this department should be sent to Roy Dubisch, Department of Mathematics, University of Washington, Seattle, Washington.

ON THE SOLUTION OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

ANNE RESEIGH CLOUGH AND DAVID DICKINSON,
University of Massachusetts

A standard way of solving linear homogeneous differential equations with polynomial coefficients is the series method of Frobenius. The differential equations of the generalized hypergeometric functions, when solved by this method, have non logarithmic series solutions whose coefficients are found directly by solving a two-term recurrence relation as well as logarithmic solutions that are derivable from a two-term recurrence relation. We are here interested in a converse of this statement:

THEOREM: *If the non-logarithmic solutions of a linear homogeneous differential equation with polynomial coefficients are series whose coefficients are solutions of a two-term recurrence relation, then the differential equation is essentially a generalized hypergeometric differential equation and the solutions are generalized hypergeometric series or logarithmic solutions associated with generalized hypergeometric series.*

This theorem is well-known to workers in the special functions of analysis but since it has seemed to be such a surprising result to others, we believe that this theorem should be made a matter of public record. Recurrence relations with more than two terms are so difficult to solve explicitly that virtually all the examples of the Frobenius method given in the standard textbooks that yield series with explicitly given coefficients are examples covered by this theorem.

Our analysis will rest heavily on the use of the operator $\theta = x(d/dx)$. It is a linear operator having the properties $\theta x^n = nx^n$ and $\theta[\theta^n]f(x) = \theta^{n+1}f(x)$.

Let $\{P_k(\theta)\}_{k=0}^{n}$ be a set of polynomials in θ with coefficients independent of x and such that the polynomials of greatest degree have degree precisely t. Then the most general linear homogeneous differential equation with polynomial coefficients of degree t can be written

$$\left[\sum_{k=0}^{n} x^k P_k(\theta)\right] y = 0. \tag{1}$$

Let us first assume the possibility of a solution of (1) of the form

$$y = \sum_{i=0}^{\infty} a_i x^{i+c}, \qquad a_0 \neq 0.$$

Substituting this series into (1), we obtain

$$\left[\sum_{k=0}^{n} x^k P_k(\theta)\right] \sum_{i=0}^{\infty} a_i x^{i+c} = \sum_{k=0}^{n} \sum_{i=0}^{\infty} a_i P_k(i+c) x^{i+c} = 0.$$

Shifting the index i to $i-k$, we obtain

$$\sum_{k=0}^{n}\sum_{i=k}^{\infty} a_{i-k}P_k(i-k+c)x^{i-k+c} = 0.$$

Since all the coefficients of this power series must vanish, we have, setting the coefficients equal to zero,

$$\sum_{k=0}^{n} a_{i-k}P_k(i-k+c) = 0, \qquad i \geqq n.$$

By hypothesis, this $(n+1)$ term recurrence relation only has two terms. That is, it is actually of the form

$$a_{i-s}P_s(i-s+c) + a_{i-t}P_t(i-t+c) = 0,$$

and the equation (1) of order t could have been written as

$$[x^sP_s(\theta) + x^tP_t(\theta)]y = 0$$

or

$$(2) \qquad [x^rP_s(\theta) + P_t(\theta)]y = 0$$

where we may assume that $P_t(\theta)$ is of degree precisely t and that s, the degree of $P_s(\theta)$, does not exceed t. We shall not assume that r is a positive integer—only that it is a non-zero number. By a change of independent variable, (2) may be put into one form of the generalized hypergeometric equation. However, our interest here is within the framework of the usual Frobenius method and we shall solve (2) directly.

Now let us assume the possibility of a solution of (2) of the form

$$(3) \qquad y = \sum_{i=0}^{\infty} a_i x^{ir+c}, \qquad a_0 \neq 0.$$

Note that this is a descending series if $r<0$ and an ascending series if $r>0$.

Substituting this series into (2), we obtain

$$(4) \qquad \begin{aligned} 0 = [x^rP_s(\theta) + P_t(\theta)]\sum_{i=0}^{\infty} a_i x^{ir+c} &= \sum_{i=0}^{\infty} a_iP_s(ir+c)x^{ir+r+c} \\ &+ \sum_{i=0}^{\infty} a_iP_t(ir+c)x^{ir+c}. \end{aligned}$$

We now shift the index i to $i-1$ in the first summation of the rightmost member of (4) and thus obtain

$$0 = \sum_{i=1}^{\infty} a_{i-1}P_s(ir-r+c)x^{ir+c} + \sum_{i=0}^{\infty} a_iP_t(ir+c)x^{ir+c}.$$

If we now equate to zero the coefficients of equal powers of x, we find that

$$a_0 P_t(c) = 0 \tag{5}$$

and, for $i>0$,

$$a_i = -\frac{P_s(ir - r + c)}{P_t(ir + c)}\, a_{i-1}. \tag{6}$$

In the non-exceptional case we are considering, the indicial equation (5) has t distinct roots and each value of c that is a root of (5) gives rise to a recurrence relation (6) and one solution of our equation of order t. If (5) has some roots that differ by an integral multiple of r (i.e., for some i, a denominator of (6) vanishes) or if (5) has multiple roots, the two-term recurrence relation (6) will not furnish us directly the coefficients of all t solutions. The other solutions will be logarithmic.

By iterating (6), we find that

$$a_i = (-1)^i \frac{\prod_{j=1}^{i} P_s(jr - r + c)}{\prod_{j=1}^{i} P_t(jr + c)}\, a_0, \qquad i > 0. \tag{7}$$

Let us now write the polynomials in θ in their factored form:
$P_s(\theta) = a\prod_{k=1}^{s} (\theta - S_k)$, $P_t(\theta) = b\prod_{k=1}^{t} (\theta - T_k)$. The expression (7) for the series coefficients then becomes

$$a_i = (-1)^i \frac{\prod_{j=1}^{i} a \prod_{k=1}^{s} (jr - r + c - S_k)}{\prod_{j=1}^{i} b \prod_{k=1}^{t} (jr + c - T_k)}\, a_0$$

or

$$a_i = \left(\frac{-a}{b}\right)^i \frac{r^{si} \prod_{k=1}^{s} \prod_{j=1}^{i} \left[j - 1 + \frac{c - S_k}{r}\right]}{r^{ti} \prod_{k=1}^{t} \prod_{j=1}^{i} \left[j + \frac{c - T_k}{r}\right]}\, a_0, \qquad i > 0. \tag{8}$$

The factorial function is defined for i a non-negative integer as

$$(a)_i = (a)(a + 1)(a + 2) \cdots (a + i - 1) = \frac{\Gamma(a + i)}{\Gamma(a)}\,. \tag{9}$$

Hence (8) may be written

$$a_i = \left(\frac{-a}{b}\right)^i \frac{r^{(s-t)i} \prod_{k=1}^{s} \left(\frac{c - S_k}{r}\right)_i}{\prod_{k=1}^{t} \left(\frac{c - T_k}{r} + 1\right)_i}\, a_0, \qquad i \geqq 0$$

and from (3), the solutions of (2) are

$$(10)\qquad y = a_0x^c \sum_{i=0}^{\infty} \frac{\prod_{k=1}^{s}\left(\frac{c-S_k}{r}\right)_i}{\prod_{k=1}^{t}\left(\frac{c-T_k}{r}+1\right)_i}\left(\frac{-ar^{s-t}x^r}{b}\right)^i$$

there being, in the non-logarithmic case, one solution for each of the t values of c that satisfy (5). In the notation for the generalized hypergeometric series, the solutions of (2) are then

$$(11)\qquad y = x^c{}_{s+1}F_t\left[\begin{matrix}\frac{c-S_1}{r}, \frac{c-S_2}{r}, \cdots, \frac{c-S_s}{r}, 1; & \frac{-ar^{s-t}x^r}{b}\\ \frac{c-T_1}{r}+1, \cdots, \frac{c-T_t}{r}+1; & \end{matrix}\right]$$

If (2) has logarithmic solutions, they may be derived by standard methods from (10).

Differential equations that suit the hypothesis of our theorem are quite easy to recognize. They are of the form

$$(12)\qquad \sum_{i=0}^{t} (A_ix^r + B_i)x^iy^{(i)} = 0$$

where not all of the A_i and not all of the B_i are zero. Such an equation can be converted quickly into the θ form by use of the identity

$$x^iy^{(i)} = (\theta)(\theta-1)(\theta-2)\cdots(\theta-i+1)y.$$

For an example, we choose Bessel's equation of order α:

$$x^2y'' + xy' + (x^2-\alpha^2)y = 0$$

or

$$(0x^2+1)x^2y'' + (0x^2+1)xy' + (1x^2-\alpha^2)y = 0.$$

In the θ form it is $[x^2+(\theta^2-\alpha^2)]y=0$, and the indicial equation $c^2-\alpha^2=0$ has the roots $\alpha\geqq 0$ and $-\alpha$.

From (10) and (11), one solution is

$$y_1 = x^\alpha \sum_{i=0}^{\infty} \frac{1}{\left(\frac{\alpha+\alpha}{2}+1\right)_i\left(\frac{\alpha-\alpha}{2}+1\right)_i}\left(\frac{-x^2}{4}\right)^i = x^\alpha{}_0F_1(-;\alpha+1;-x^2/4)$$

$$= 2^\alpha\Gamma(\alpha+1)J_\alpha(x)$$

and the other solution is

$$y_2 = x^{-\alpha} \sum_{i=0}^{\infty} \frac{1}{(-\alpha + 1)_i i!} \left(\frac{-x^2}{4}\right)^i = x^{-\alpha}{}_0F_1(-; -\alpha + 1; -x^2/4)$$
$$= 2^{-\alpha}\Gamma(-\alpha + 1)J_{-\alpha}(x)$$

unless $-\alpha$ is zero or a negative integer. If $-\alpha$ is zero, this second solution is not independent of the first and if $-\alpha$ is a negative integer, this second solution above does not exist. In these two cases, a logarithmic solution must be found.

PLANE GEOMETRY AND COMPLEX NUMBERS

ROBERT G. BLAKE, University of Florida

Let the complex number $z = x + iy$ represent a point in the Euclidean plane, $d = (x^2 + y^2)^{1/2}$ its distance from the origin, and $\lambda = (x + iy)/d$ the direction of its radius vector. Since multiplying a complex number by i has the effect of rotating its radius vector through an angle of $\pi/2$, $i\lambda$ is perpendicular to λ. Then if b is any real number, the complex number $z = i\lambda b$ represents a point on a line through the origin and perpendicular to λ. Every point on the line through the origin and perpendicular to λ can be represented by $z = i\lambda b$. Then the complex conjugate of z is given by $\bar{z} = -i\bar{\lambda}b$. Eliminating b between these two equations gives as the equation of the line through the origin perpendicular to λ:

$$\bar{\lambda}z + \lambda\bar{z} = 0. \tag{1}$$

Translating to a new origin z_0 gives as the equation of a line through the point z_0 and perpendicular to λ:

$$\bar{\lambda}z + \lambda\bar{z} = \bar{\lambda}z_0 + \lambda\bar{z}_0. \tag{2}$$

If the distance from the origin to the line is d, the line will pass through the point $z_0 = \lambda d$. Substituting this in the above equation gives as the normal form of the equation of a line:

$$\bar{\lambda}z + \lambda\bar{z} = 2d. \tag{3}$$

If the line is to pass through the points z_1 and z_2 we have

$$\bar{\lambda}z_1 + \lambda\bar{z}_1 = 2d \quad \text{and} \quad \bar{\lambda}z_2 + \lambda\bar{z}_2 = 2d.$$

Solving these for λ and $2d$ gives

$$\lambda = \frac{i(z_2 - z_1)}{|z_2 - z_1|}, \qquad 2d = \frac{i(z_2\bar{z}_1 - z_1\bar{z}_2)}{|z_2 - z_1|}.$$

A triangle with 0, z_1, z_2 as vertices has $|z_2 - z_1|$ as base and d as altitude. Therefore its area is given by

$$\text{Area Tri } (0, z_1, z_2) = i(z_2\bar{z}_1 - z_1\bar{z}_2)/4$$

Since Area Tri (z_1, z_2, z_3) = Area Tri $(0, z_1, z_2)$ + Area Tri $(0, z_2, z_3)$ − Area Tri $(0, z_1, z_3)$ we have

(4) $$\text{Area Tri}\,(z_1, z_2, z_3) = (i/4)\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}.$$

From equation (4) it follows that the equation of a line through z_1 and z_2 can be written:

(5) $$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

If $|z| = a$, then z lies on a circle of radius a with the origin as center. Since $z\bar{z} = |z|^2$, the equation of a circle of radius a with center at the origin can be written:

(6) $$z\bar{z} = a^2.$$

Translating to a new origin, z_0, gives as the equation of a circle of radius a with center at z_0:

(7) $$z\bar{z} - \bar{z}_0 z - z_0\bar{z} + z_0\bar{z}_0 = a^2.$$

If the circle passes through the origin, $z_0\bar{z}_0 = a^2$, and its equation is

(8) $$z\bar{z} - \bar{z}_0 z - z_0\bar{z} = 0.$$

If the points z_1, z_2, z_3 are not in a straight line then it follows from equation (4) that

$$\Delta = \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} \neq 0.$$

In this case expanding the determinant in the equation

(9) $$\begin{vmatrix} z\bar{z} & z & \bar{z} & 1 \\ z_1\bar{z}_1 & z_1 & \bar{z}_1 & 1 \\ z_2\bar{z}_2 & z_2 & \bar{z}_2 & 1 \\ z_3\bar{z}_3 & z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0,$$

dividing by Δ and putting

$$z_0 = -(1/\Delta)\begin{vmatrix} z_1\bar{z}_1 & z_1 & 1 \\ z_2\bar{z}_2 & z_2 & 1 \\ z_3\bar{z}_3 & z_3 & 1 \end{vmatrix} \quad \text{and} \quad z_4 = (1/\Delta)\begin{vmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ z_3^2 & z_3 & 1 \end{vmatrix}$$

gives the equation

$$z\bar{z} - \bar{z}_0 z - z_0\bar{z} + z_0\bar{z}_0 = z_4\bar{z}_4.$$

Thus equation (9) is the equation of a circle with the center at z_0 and a radius of $|z_4|$.

These equations can be used to study certain transformations. For example, substituting $z=1/w$ in equation (1) gives $\lambda w+\bar\lambda\bar w=0$ showing that the transformation $z=1/w$ carries a line through the origin into a conjugate line through the origin. Making the same substitution in equation (3) gives $w\bar w-(\lambda/2d)w-(\bar\lambda/2d)\bar w=0$ showing that a line not through the origin is transformed into a circle through the origin. Substituting in equation (7) gives

$$w\bar w - [z_0/(z_0\bar z_0 - a^2)]w - [z_0/(z_0\bar z_0 - a^2)]\bar w + 1/(z_0\bar z_0 - a^2) = 0.$$

This is the equation of a circle with its center at $\bar z_0/(z_0\bar z_0-a^2)$ and a radius of $a/|z_0\bar z_0-a^2|$. Substituting in equation (8) gives

$$(z_0/|z_0|)w + (\bar z_0/|z_0|)\bar w = 1/|z_0|$$

showing that a circle through the origin is transformed into a line not passing through the origin.

The distance from the point z to the line $\bar\lambda z'+\lambda\bar z'=2d$ is $|d-(\bar\lambda z+\lambda\bar z)/2|$. Therefore the equation of a conic section with $\lambda z'+\lambda\bar z'=2d$ as directrix, focus at z_0, and eccentricity e is:

$$|z - z_0| = e|d - (\bar\lambda z + \lambda\bar z)/2|.$$

This can be written:

$$e^2\bar\lambda^2z^2 + (2e^2 - 4)z\bar z + e^2\lambda^2\bar z^2 + 4(\bar z_0 - de^2\bar\lambda)z + 4(z_0 - de^2\lambda)\bar z + 4(d^2e^2 - z_0\bar z_0) = 0. \tag{10}$$

The substitution

$$z = \lambda z + \lambda d + \lambda(\bar\lambda z_0 - \lambda\bar z_0)/2 \tag{11}$$

will rotate and translate the figure so that the imaginary axis is the directrix and the focus is on the real axis at the point $z_0=\bar z_0=(\bar\lambda z_0+\lambda\bar z_0)/2-d$. The resulting equation is:

$$e^2z^2 + (2e^2 - 4)z\bar z + e^2\bar z^2 + 4z_0(z + \bar z) - 4z_0^2 = 0. \tag{12}$$

The vertices will then lie on the real axis and if the vertex is z_v then $z_v=\bar z_v$. Substituting this in equation (12) gives

$$(1 - e^2)z_v^2 - 2z_0z_v + z_0^2 = 0.$$

If $e\neq 1$ this has two solutions

$$z_v = z_0/(1 + e) \quad\text{or}\quad z_v = z_0/(1 - e).$$

From these we see that the center is at $z_c=z_0/(1-e^2)$ and that the length of the axis is $|2ez_0/(1-e^2)|$.

References

H. Kober, Dictionary of conformal representations. Dover Publications, Inc. New York (1952).

Ruel V. Churchill, Complex variables and applications, Chapter 1. McGraw-Hill Book Co., Inc. New York. Second Ed. (1960).

Walter B. Carver, The conjugate coordinate system for plane Euclidean geometry, Slaught Memorial Paper, Number 5, Amer. Math. Monthly, **63**, No. 9, (November 1956).

O THOU LEAST SQUARE

As we pursue our complex course
Our anchor, origin, and source
Now and forever rests in thee—
And we would keep divisor-free
    Thy additive identity.

The span to thee remains our norm.
Thy real and pure-imagined form
Can never a divisor be—
So keep us from the heresy
    Of trying to divide by thee.

MARLOW SHOLANDER

COMMENTS ON PAPERS AND BOOKS

EDITED BY HOLBROOK M. MACNEILLE, Case Institute of Technology

This department will present comments on papers published in the MATHEMATICS MAGAZINE, lists of new books, and reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent to Holbrook M. MacNeille, Department of Mathematics, Case Institute of Technology, Cleveland 6, Ohio.

ON A FORMULA FOR COS nx

ANDRZEJ MAKOWSKI, Warsaw, Poland

In May–June 1961 issue of the MATHEMATICS MAGAZINE (p. 271–273) Edgar Karst gave a method to find the coefficients of cos nx in terms of $\cos^k x$. His rule is based on some observation and is not strictly proved.

Below is given the method based on the theory of Tchebicheff polynomials.

The n-th Tchebicheff polynomial $T_n(t)$ is defined by formula

$$T_n(t) = \frac{1}{2^{n-1}} \cos (n \text{ arc} \cos t).$$

It is a polynomial of the n-th degree. Putting $t=\cos x$ we get $2^{n-1}T_n(\cos x)=\cos nx$. Thus the problem discussed by E. Karst is equivalent to that to find the coefficients of $f_n(t)=2^{n-1}T_n(t)$. The following formula is known [1]

$$T_{n+1}(t) - tT_n(t) + \frac{1}{4} T_{n-1}(t) \equiv 0.$$

Multiplying both sides by 2^n we get

$$f_{n+1}(t) - 2tf_n(t) + f_{n-1}(t) \equiv 0.$$

which is equivalent to the formula

$$\cos (n+1)x - 2 \cos x \cos nx + \cos (n-1)x \equiv 0.$$

The last formula may be easily verified.

Let

$$f_{n+1}(t) = \sum_{i=0}^{n+1} a_i t^i, \qquad f_n(t) = \sum_{j=0}^{n} b_j t^j, \qquad f_{n-1}(t) = \sum_{i=0}^{n-1} c_i t^i.$$

Hence

$$\sum_{i=0}^{n+1} a_i t^i - 2 \sum_{j=0}^{n} b_j t^{j+1} + \sum_{i=0}^{n-1} c_i t^i \equiv 0.$$

We put $c_{n+1}=c_n=b_{-1}=0$. Thus

$$\sum_{i=0}^{n+1} a_i t^i - 2 \sum_{i=0}^{n+1} b_{i-1} t^i + \sum_{i=0}^{n+1} c_i t^i \equiv 0$$

and

$$\sum_{i=0}^{n+1} (a_i - 2b_{i-1} + c_i) t^i \equiv 0.$$

Hence $a_i - 2b_{i-1} + c_i = 0$ and $a_i = 2b_{i-1} - c_i$ for $i = 0, 1, 2, \cdots, n+1$. Now we can obtain the coefficients of $f_n(t)$ similarly as those in the binomial formula (Pascal's triangle).

	a_{-1}	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	0	0	1					
2	0	−1	0	2				
3	0	0	−3	0	4			
4	0	1	0	−8	0	8		
5	0	0	5	0	−20	0	16	
6	0	−1	0	18	0	−48	0	32

The column a_{-1} consists of zeros. In the first row we write 0, 1 (since $\cos x = 0 + 1 \cos x$), in the second −1, 0, 2 (since $\cos 2x = -1 + 0 \cos x - 2 \cos^2 x$). If there are written down $k \geqq 2$ rows, any element of the next row is equal to a difference between twice the element standing one row up and one column left and the element standing two rows up.

From the sixth row of the table we find

$$\cos 6x = -1 + 18 \cos^2 x - 48 \cos^4 x + 32 \cos^6 x.$$

Reference

1. S. Kaczmarz, H. Steinhaus, Theorie der Orthogonalreihen, New York (1951), 115–116.

BOOK REVIEWS

Understanding Basic Mathematics. By L. H. Miller. Holt, Rinehart and Winston, New York, 1961, x+499 pages. $6.25.

Fundamental Mathematics. By T. L. Wade and H. E. Taylor. McGraw-Hill, 2nd edition, 1961, xiv+428 pages. $6.75.

As the scope and purpose of these two books are generally the same, it is convenient to review them together. Both aim to provide a review of elementary mathematics which can be used as a college text. The main differences lie in the details of the topics covered. Miller has a chapter on Geometry, which hardly does justice even to the elements of the subject (the reader is advised to consult a reference book for definitions of most geometrical terms), and another chapter on Analytical Geometry. There is also a chapter devoted to a selection of subjects, such as number bases, clock numbers and automatic digital computation, which have only recently begun to find their way into college courses. While the appearance of these subjects is welcome, I think their separation is not so welcome as it may foster the notion that they are mathematically, and not merely academically, different from the main part of the course. Wade and Taylor do not cover all these subjects, but where they are introduced it is in their natural place in the text, for example, the section on number bases is in the introductory chapter as part of the discussion on methods of counting.

Wade and Taylor devote a chapter to statistics, a subject not mentioned by Miller.

On the whole the presentation of Wade and Taylor is better, and they have had the advantage of this second edition to improve on the first. There are in Miller's book more errors in the type, and some oddities, such as the definition of an even number as one ending in 0, 2, 4, 6 or 8, and the example which reads "If $3p^2=q^2$, find a factor of q," the implication of which, that $q=0$, may not be realised. Both books are well provided with exercises. If a choice has to be made between them, the differences in the subjects covered are likely to be more important than any differences in presentation.

ALAN SUTCLIFFE
Knottingley, Yorkshire, England

Applied Boolean Algebra, An Elementary Introduction, by Franz E. Hohn. Macmillan Company, New York. 139 pages, $2.50. (paper)

This is one of the few books on elementary Boolean algebra and its applications to basic switching logic. A good list of references is given at the end of each chapter.

As the author indicates, the book is self-contained and requires little engineering or mathematical background, but it does require a reasonable amount of mathematical maturity. The text and examples are very good, but the extent of material covered is sometimes too brief, and a better continuity between chapters could be arranged. It is required that the reader fill in this continuity. The book covers analysis and application of Boolean algebra to combinational relay circuity, propositional logic, and the algebra of sets which are all written in a parallel axiomatic form. There is an excellent selection of problems at the end of each section.

The last sections on the map problem and the semiconductor logic elements are too brief for the average inexperienced reader.

This book will make a good text for introduction to switching logic and Boolean algebra applications.

MILLARD T. BATTLES
Boeing Co.

The Pentagon. Published semiannually by Kappa Mu Epsilon. Edited by Fred W. Lott, Jr., State College of Iowa, Cedar Falls, Iowa. $2.00 for two years.

The Pentagon is a mathematics magazine for students and the official publication of Kappa Mu Epsilon, a national honorary mathematics fraternity with 63 active chapters at colleges distributed throughout 23 states. As might be expected, about 10 of the 60 pages in each issue are devoted to fraternity news. The remainder of the magazine contains The Problem Corner, The Mathematical Scrapbook, The Book Shelf, and articles by students and faculty. The problems are at the undergraduate level, the short quotations in the scrapbook are nicely chosen from a wide variety of sources, and the books reviewed are those likely to interest students.

The student articles are carefully selected and edited and are by no means trivial. Recent titles include: Cryptography with Matrices, Graphical Integrations, An Analysis of Some of the Syllogisms Found in Alice in Wonderland, Quadrispace, An Adventure with Spirals, Areas of Simple Polygons, The Nomogram. The faculty articles are of types interesting to undergraduate students.

The format of the letter-press printed, 6″×9″ magazine is pleasing. Copies of *The Pentagon* on the magazine shelves of any college should prove stimulating to its mathematics students. At 50 cents a copy, why should any librarian hesitate?

CHARLES W. TRIGG
Los Angeles City College

BOOKS RECEIVED FOR REVIEW

Banach Spaces of Analytic Functions. By K. Hoffman, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, xiii+217 pages, $8.00.

Studies in Mathematics, Volume I. MAA Studies in Modern Analysis. By E. J. McShane, M. H. Stone, E. R. Lorch, and Casper Goffman, The Mathematical Association of America, Buffalo, 1962, vii+182 pages, $4.00.

Spectral Theory. By Edgar R. Lorch, Oxford, New York, 1962, iv+158 pages, $5.50.

Topology of 3-Manifolds and Related Topics. Edited by M. K. Fort, Jr., Prentice-Hall, Englewood Cliffs, New Jersey, 1962, ix+256 pages, $7.50.

Elementary Concepts of Topology. By Paul Alexandroff, Dover, New York, 1961, 15+73 pages, $1.00 (paper).

Lebesgue Integration. By J. H. Williamson, Holt, Rinehart and Winston, New York, 1962, v+117 pages. $3.25.

Inequalities. By P. P. Korovkin, Blaisdell, New York, 1961, v+60 pages, 95¢ (paper).

The Method of Mathematical Induction. By I. S. Sominskii, Blaisdell, New York, 1961, v+56 pages, 95¢ (paper).

The Real Number System. By John M. H. Olmsted, Appleton-Century-Crofts, New York, 1962, viii+216 pages, $3.95.

The Real Number System in an Algebraic Setting. By J. B. Roberts, W. H. Freeman, San Francisco and London, 1962, ix+145 pages, $1.75 (paper).

Fibonacci Numbers. By N. N. Vorob'ev, Blaisdell, New York, 1961, v+66 pages, 95¢ (paper).

The USSR Olympiad Problem Book. By D. O. Shklarsky, N. N. Chentzov and I. M. Yaglom, W. H. Freeman, San Francisco, 1962, xvi+452 pages, $9.00.

Charles Babbage and His Calculating Machine. By Charles Babbage, Dover, New York, 1961, vii+400 pages, $2.00 (paper).

Analogue Computation. By R. W. Williams, Academic Press, New York, 1962, 271 pages, $9.50.

An Introduction to Ordinary Differential Equations. By Earl A. Coddington, Prentice-Hall, Englewood Cliffs, New Jersey, 1961, viii+292 pages, $6.00.

Elementary Differential Equations, Second Edition. By William T. Martin and Eric Reissner, Addison-Wesley, Reading. Mass., 1961, xiii+331 pages, $6.75.

Differential Equations. By Kaj L. Nielsen, Barnes & Noble, New York, xxii+262 pages, $1.75 (paper).

Differential Equations. By C. W. Leininger, Harper, New York, 1962, x+271 pages, $6.00.

An Introduction to Probability and Mathematical Statistics. By Howard G. Tucker, Academic Press, New York, 1962, xii+228 pages, $5.75.

Introduction to Probability and Statistics. By Henry L. Alder and Edward B. Roessler, Second Edition, W. H. Freeman, San Francisco and London, 1962, 289 pages, $5.50.

The Ruler in Geometrical Constructions. By A. S. Smogorzhevskii, Blaisdell, New York, 1961, v+86 pages, 95¢ (paper).
Geometrical Constructions Using Compasses Only. By A. N. Kostovskii, Blaisdell, New York, 1961, v+78 pages, 95¢ (paper).
Revision Geometry. By J. O. Oyelese, Cambridge, New York, 1962, v+296 pages, $2.00.
The Fourth Dimension Simply Explained. By Henry P. Manning, Dover, New York, 1960, iii+251 pages, $1.35.
Analytic Geometry: A Vector Approach. By Charles Wexler, Addison-Wesley, Reading, Mass., 1962, x+291 pages, $6.00.
Analytic Geometry, Second Edition. By Gordon Fuller, Addison-Wesley, Reading, Mass., 1962, 230 pages, $5.75.
Descriptive Geometry. By C. E. Douglass and A. L. Hoag, Holt, Rinehart and Winston, New York, 1962, 197 pages, $5.00.
Introduction to Mathematical Analysis. By R. E. Johnson, N. H. McCoy and A. F. O'Neill, Holt, Rinehart and Winston, New York, 1962, xiii+476 pages, $7.00.
Introductory College Mathematics. By Chester G. Jaeger and Harold M. Bacon, Second Edition, Harper, New York, 1954, xvii+423 pages, $6.50.
Basic Analysis. By Stephen P. Hoffman, Holt, Rinehart and Winston, New York, 1961, xii+459 pages, $6.50.
Introduction to Modern Algebra and Analysis. By Ralph Crouch and Elbert Walker, Holt, Rinehart and Winston, New York, 1962, xiii+152 pages, $4.00.
Elementary Mathematical Analysis. By A. E. Labarre, Addison-Wesley, Reading, Mass., 1961, v+706 pages, $7.75.
Differential and Integral Calculus. By Clyde E. Love and Earl D. Rainville, Macmillan, New York, 1962, xvii+579 pages, $7.50.
Calculus, Second Edition. By George B. Thomas, Addison-Wesley, Reading, Mass., 1961, xiii+850 pages, $8.75.
Capsule Calculus. By Ira Ritow, Doubleday and Company, New York, 1962, xiii+177 pages, $1.45 (paper).
Basic Matrix Theory. By L. Fuller, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, ix+245 pages, $7.00.
Topics in Modern Algebra. By Charles P. Benner, Albert Newhouse, Cortez B. Rader and Richard L. Yates, Harper, New York, 1962, x+144 pages, $3.75.
Modern Algebra, First Course. By R. E. Johnson, L. L. Lendsey, W. E. Slesnik, Addison-Wesley, Reading, Mass., 1961, iii+628 pages, $5.28.
Modern Algebra: Structure and Method, Book I. By Mary P. Dolciani, Simon L. Berman, Julius Freilich, Houghton Mifflin, Boston, 1962, v+552 pages, $4.68.
Modern College Algebra. By Elbridge P. Vance, Addison-Wesley, Reading, Mass., v+260 pages, 1962, $5.75.
College Algebra. By Charles H. Lehmann, John Wiley, New York, 1962, xi+432 pages, $5.95.
College Algebra. By Max Peters, Barron's Educational Series, Great Neck, New York, 1962, 694 pages, $2.25 (paper), $4.50 (cloth).
Introductory Algebra and Trigonometry. By A. Spitzbart and R. H. Bardell, Addison-Wesley, Reading, Mass., 1962, x+389 pages, $7.50.
Algebra and Trigonometry. By Irwin Miller and Simon Green, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, x+358 pages, $6.95.
Modern Algebra and Trigonometry. By E. P. Vance, Addison-Wessley, Reading, Mass., 1962, v+374 pages, $7.50.
Functional Trigonometry. By Abraham P. Hillman and Gerald L. Alexanderson, Allyn and Bacon, Boston, 1961, v+327 pages, $5.95.
Elementary and Advanced Trigonometry. By Kenneth S. Miller and John B. Walsh, Harper, New York, 1962, xi+350 pages, $5.75.
Unified Algebra and Trigonometry. By Dick Wick Hall and Louis O. Kattsoff, John Wiley, New York, 1962, xii+455 pages, $6.75.

Physics in the Soviet Union. By A. S. Kompanayets, Philosophical Library, New York, 1962, 7+592 pages, $7.50.

Forces and Fields. By Mary B. Hesse, Philosophical Library, New York, 1961. x+318 pages, $10.00.

Fundamentals of Celestial Mechanics. By J. M. A. Danby, Macmillan, New York, 1962, xiii+348 pages, $8.00.

Archimedes and the Door of Science. By Jeanne Bendick, Franklin Watts, New York, 1962, v+143 pages, $1.95.

Oscillatory Motions. By Jules Haag, Translated by Reinhardt M. Rosenberg, Wadsworth Publishing Co., Belmont, Calif., 1962, xxi+201 pages, technical edition $11.50, Trade edition: $15.35.

Some Applications of Mechanics to Mathematics. By V. A. Uspenskii, Blaisdell, New York, 1961, v+58 pages, 95¢ (paper).

Algebra with Applications to Business Economics. By Paul H. Daus and William M. Whyburn, Addison-Wesley, Reading, Mass., 1961, v+354 pages, $6.75.

Reliability: Management, Methods, and Mathematics. By David K. Lloyd and Myron Lipow, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, xxii+528 pages, $11.25.

Treasury of World Science. Edited by Dagobert D. Runes, Philosophical Library, New York, 1962, xxi+978 pages, $15.00.

The Teaching of Arithmetic. By F. F. Potter, Philosophical Library, New York, 1961, v+462 pages, $4.75.

Teaching Elementary Arithmetic. By C. B. Thorpe, Harper, New York, 1962, ix+412 pages, $6.50.

Mathematics for Secondary School Teachers. By Bruce E. Meserve and Max A. Sobel, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, xi+367 pages, $6.75.

Introduction to Mathematics. By Charles F. Brumfiel, Robert E. Eicholz, Merril E. Shanks, Addison-Wesley, Reading, Mass., 1961, xi+323 pages, $4.00.

Fundamental Concepts of Elementary Mathematics. By C. F. Brumfiel, R. E. Eicholz and M. E. Shanks, Addison-Wesley, 1962, v+340 pages, $5.75.

An Introduction to the Elements of Mathematics. By John N. Fujii, John Wiley, New York, 1961, v+312 pages, $6.25.

Arithmetic: An Introduction to Mathematics. By L. Clark Lay, Macmillan, New York, 1961, xiii+323 pages, $4.50.

Arithmetic for College Students, Revised Edition. By Edwin I. Stein, Allyn and Bacon, Boston, 1961, xiv+450 pages, $5.50.

The New Mathematics. By Irving Adler, The New American Library of World Literature, New York, 1962, 192 pages, 60¢ (pocket).

Basic Concepts in Modern Mathematics. By John E. Hafstrom, Addison-Wesley, Reading, Mass., 1961, vii+195 pages, $5.75.

Complete Slide Rule Handbook. By J. N. Arnold, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, viii+206 pages, $2.45 (paper).

Six-Figure Logarithms, Antilogarithms and Logarithmic Trigonometrical Functions. By C. Attwood, Pergamon Press, New York, 1961, 139 pages, $2.00 (paper).

Mathematics for Practical Use. By Kaj L. Nielsen, Barnes and Noble, New York, 1962, xiii+237 pages, $1.25 (paper).

Mathematics in Everyday Things. By William C. Vergara, The New American Library of World Literature, New York, 1962, 300 pages, 75¢ (pocket).

Study and Succeed. By Lyle Tussing, John Wiley, New York, 1962, ix+157 pages, $2.95 (paper).

Drafting Made Simple. By Yonny Segel, Doubleday, New York, 1962, 192 pages, $1.45 (paper).

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.

PROPOSALS

488. *Proposed by Josef Andersson, Vaxholm, Sweden.*

For what values of n is the sum of n integral cubed primes a square (n positive and odd)?

489. *Proposed by Dewey Duncan, East Los Angeles College, California.*

In Archimedes' most prized theorem which was depicted on his tombstone [described by Cicero when serving in Sicily], the common ratio of total areas and volumes of a right circular cylinder to the corresponding entities of the inscribed sphere is 2/3. Replace the cylinder by a right circular cone having its vertex a right angle. Show that the ratio arising here is $(2/3)^2$.

490. *Proposed by David L. Silverman, Beverly Hills, California.*

There are twice as many routes from Kent to Ghent, including those through Trent, as there are from Kent to Trent, including those through Ghent. On the other hand, there are twice as many direct routes from Ghent to Trent as there are from Kent to Ghent. How many direct routes connect each pair of towns?

491. *Proposed by C. D. Smith, University of Alabama.*

Given the isosceles triangle with sides a, a, b. Draw altitude h to side b. The in-radius of the given triangle is r, and r_1 is the in-radius of each triangle formed by h. Prove that the in-circle (r) is greater than, equal to, or less than the sum of the two circles (r_1) when $b \lesseqgtr a\sqrt{2}$.

492. *Proposed by Ronald Butler, University of Saskatchewan.*

Given n points in a cartesian coordinate plane. Obtain a ruler and compass construction for determining a point, of arbitrary given abscissa, lying on the $(n-1)$st degree curve through the given points.

493. *Proposed by Andrzej Makowski, Warsaw, Poland.*

Prove that $n^4+4^n (n=1, 2, 3 \cdots)$ is a prime number only for $n=1$.

494. *Proposed by C. W. Trigg, Los Angeles City College.*

What two rational fractions with denominators less than 100 most closely approximate $\sqrt[3]{98}$ by excess and by defect?

SOLUTIONS

Late Solutions

425, 457. *Andrzej Makowski, Warsaw, Poland.*

451, 455, 456. *Josef Andersson, Vaxholm, Sweden; W. C. Waterhouse, Harvard University.*

458. *Josef Andersson, Vaxholm, Sweden; Andrzej Makowski, Warsaw, Poland.*

460. *Huseyin Demir, Middle East Technical University, Ankara, Turkey; Gilbert Labelle, Collège de Longueuil, Canada; Andrzej Makowski, Warsaw, Poland.*

461. *Josef Andersson, Vaxholm, Sweden.*

464. *Huseyin Demir, Middle East Technical University, Ankara, Turkey; Andrzej Makowski, Warsaw, Poland.*

466. *Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

A Non-Palindromic Strobogram

467. [January 1962]. *Proposed by C. W. Trigg, Los Angeles City College.*

Identify the unique four-digit non-palindromic strobogrammatic integer which is the sum of two positive cubes. (A strobogrammatic integer reads the same after rotation through 180°, e.g. 89068.)

Solution by Prasert Na Nagara, College of Agriculture, Bangkok, Thailand. A cube is $\equiv \pm 1$, or 0 (mod 9). A sum of two cubes is $\equiv \pm 2$, ± 1, or $\equiv 0$ (mod 9). In order to be a four-digit non-palindromic strobogrammatic integer, it must contain 6, 9, and 00, 11, 69, or 88. With the use of the rule of casting out nines, we find that the only possible digits to make the integer a sum of two cubes are 1, 1, 6, and 9. Thus, we have to verify whether 1691, 1961, 6119, and 9116 are the sums of two cubes; i.e. $(3a)^3+(3b-1)^3$. Subtracting each of the numbers by $(3a)^3$ for $a<7$, and with the use of a table, we check whether the remainder is a cube. The unique solution is $6119=15^3+14^3$.

Also solved by Rodney D. Arner, Chaffey College, California; Leon Bankoff, Los Angeles, California; Merrill Barneby, University of North Dakota; Donald K. Bissonnette, Florida State University; Robert P. Boner, Kansas City, Missouri; Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts; J. L. Brown, Jr., Pennsylvania State University; Daniel I. A. Cohen, Central High School, Philadelphia, Pennsylvania; Monte Dernham, San Francisco, California; Charles E. Franti, Michigan College of Mining and Technology; Robert P. Goldberg, Cambridge, Massachusetts; Thomas R. Hamrick, San Quentin, California; Kirk Jensen, Gridley Union High School, Gridley, California; Paul Kroll, New York University; Gilbert Labelle, Collège de Longueuil, Canada; R. A. Melter, University of Missouri; Warren McCausland, Saint Mary's College, California; C. F. Pinzka, Univeristy of Cincinnati; James Shneer, George Washington University; David L. Silverman, Beverly Hills, California; Hazel S. Wilson, Jacksonville University; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer. One incorrect solution was received.

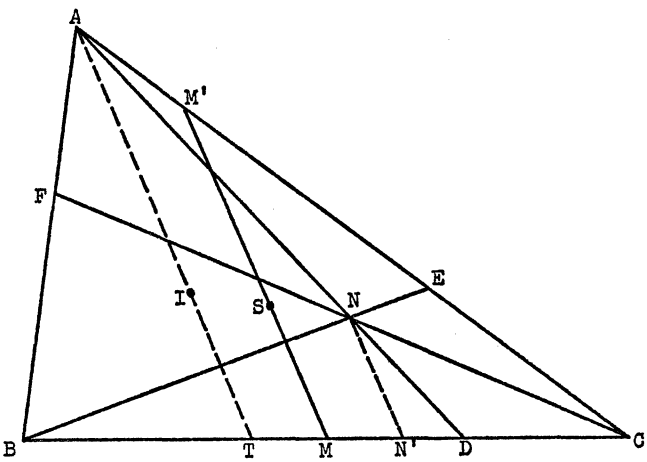
Jensen noted that $25^2+21^3=9886$.

Perimeter Bisectors

468. [January 1962]. *Proposed by Paul D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.*

The three perimeter bisecting lines, each of which passes through a corresponding midpoint of a side of a given triangle, meet at a point S. The three perimeter bisecting lines, each of which passes through a corresponding vertex of the given triangle meet in a point N. Show that S is the midpoint of the line segment joining the incenter of the given triangle to the point N.

Solution by Charles F. Pinzka, University of Cincinnati. Denote the vertices



by A, B, C, the lengths of the sides by a, b, and c, and the perimeter bisectors through the vertices by AD, BE, CF (see accompanying figure). Let MM' be the perimeter bisector through the midpoint M of BC and let AT be the bisector of angle A. Since $AE=BD=(a+b-c)/2$, $AF=CD=(a-b+c)/2$, and $BF=CE=(-a+b+c)/2$, the concurrence of AD, BE, CF follows from the converse of Ceva's theorem. Recalling that $BT:TC=c:b$, a simple calculation shows that $CM':CA=CM:CT=b+c:2b$, whence MM' is parallel to AT. Draw NN' parallel to AT, meeting BC at N'. From the theorem

$$\frac{AN}{ND}=\frac{AE}{CE}+\frac{AF}{BF},$$

we find that

$$\frac{TN'}{N'D}=\frac{AN}{ND}=\frac{2a}{-a+b+c}$$

or $TN':TD=2a:a+b+c$. A few more calculations show that $TN'=a(b-c)/(b+c)=2\,TM$. The parallels AT, MM', NN' are thus equally spaced. A similar argument on one of the other sides establishes another set of equally spaced parallels, the outer two passing through N and the incenter I. Thus N and I are opposite vertices of a parallelogram with the common point S at the center.

Also solved by Josef Andersson, Vaxholm, Sweden and the proposer.

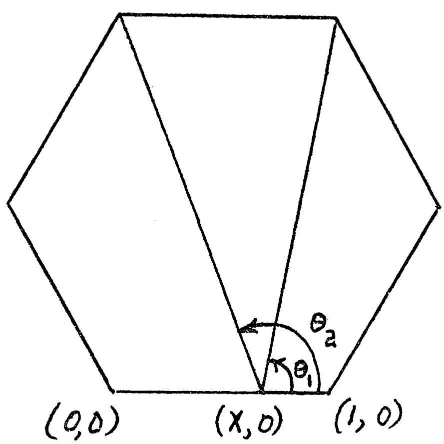
Thomas pointed out that this result is contrary to *Mathematical Reviews*, 22 (1961), 159 where it is stated that the perimeter-bisectors through the vertices of a triangle are concurrent at the incenter.

A Random Probability

469. [January 1962]. *Proposed by J. Gallego-Diaz, Universidad del Zulia, Maracaibo, Venezuela.*

A random straight line is drawn across a regular hexagon. What is the probability that it intersects two opposite sides?

I. *Solution by W. W. Funkenbusch, Michigan College of Mining and Technology.* This problem of course has multiple answers according to the particular



probability model applied to the problem. Furthermore, it is clear that any probability whatsoever ($0 \leqq p \leqq 1$) can be found by inventing the proper model. The following, however, appeals to me as being in some sense most "reasonable." Consider a regular hexagon of unit side, two vertices being (0, 0) and (1, 0). We consider (without loss of generality) that the straight line cuts the side (0, 0) to (1, 0) at $(x, 0)$, and we assume that x is uniformly distributed from 0 to 1 inclusive. Let θ be the angle made by the straight line with the side from (0, 0) to (1, 0) and we assume that θ is uniformly distributed from 0 to π inclusive. Hence the desired probability is given by

$$\text{Probability} = \frac{\int_0^1 \int_{\theta_1}^{\theta_2} d\theta dx}{\int_0^1 \int_0^{\pi} d\theta dx}$$

where $\theta = \text{Arctan } \sqrt{3}/(1-x)$ and $\theta_2 = \pi - \text{Arctan } \sqrt{3}/x$. Evaluation gives the result Probability $= \frac{1}{3} + (\sqrt{3}/\pi) \ln \frac{3}{4} = 0.175$ (approx.)

II. *Solution by Murray S. Klamkin, AVCO, Wilmington, Massachusetts.* The problem is not uniquely soluble for no definition of the random straight line distribution was given. We will obtain two answers by assuming two different distributions both of which are invariant under the group of motions in the plane (will give the same answer to all observers.) **1.** We assume that two points (which define the random line) are taken at random with a uniform distribution on the sides of the hexagon with no two points on the same side. There is no loss of generality by assuming one of the points is on a fixed side and the other is on one of the five other sides. Whence, probability of intersecting two opposite sides is 1/5. **2.** If we assume that the two points as before can also be on the same side, then probability is 1/6.

Also solved by Brother U. Alfred, Saint Mary's University, California; Leon Bankoff, Los Angeles, California; Melvin Bloom, Miami University; Gilbert Labelle, Collège de Longueuil, Canada; W. Moser, University of Manitoba; C. F. Pinzka, University of Cincinnati; and David L. Silverman, Beverly Hills, California.

A Factorial Summation

470 [January 1962]. *Proposed by William Squire, West Virginia University.*

Prove that

$$\sum_{n=0}^{N}(-1)^n\frac{(1-2N-n)!(1+N)!}{(1+n)!(N-n)!(1+N-n)!(2-2N)!}=\frac{1}{(N+2)!}\,.$$

Solution by L. Carlitz, Duke University. The formula should presumably read as follows:

$$\sum_{n=0}^{N}(-1)^n\frac{(1+2N-n)!(1+N)!}{(1+n)!(N-n)!(1+N-n)!}=\frac{(2+2N)!}{(2+N)!}\,. \tag{1}$$

The left hand side is equal to

$$\begin{aligned}\sum_{n=0}^{N}(-1)^{N-n}&\frac{(N+n+1)!(N+1)!}{n!(n+1)!(N-n+1)!}\\ &=(-1)^N(N+1)!\sum_{n=0}^{N}\frac{(N+2)_n(-N-1)_n}{n!(n+1)!}\\ &=(-1)^N(N+1)!F[N+2,-N-1;2;1]\\ &\quad-(-1)^N(N+1)!\frac{(N+2)_{N+1}(-N-1)_{N+1}}{(N+1)!(N+2)!}\,.\end{aligned}$$

By Gauss's formula

$$F[N+2,-N-1;2;1]=\frac{(-N)_{N+1}}{(2)_{N+1}}=0.$$

On the other hand

$$(-1)^{N+1}(N+1)!\frac{(N+2)_{N+1}(-N-1)_{N+1}}{(N+1)!(N+2)!}=\frac{(N+2)_{N+1}(N+1)!}{(N+2)!}$$

$$= \frac{(2N+2)!}{(N+2)!}.$$

This proves (1).

Also solved by H. W. Gould, West Virginia University; C. F. Pinzka, University of Cincinnati; and the proposer.

Pinzka found that this problem was related to Problem 186, *Mathematics Magazine*. 1953, p. 94 and 3596, *American Mathematical Monthly*, 1934, p. 56.

A Sum of Largest Integers

471 [January 1962]. *Proposed by Brother U. Alfred, St. Mary's University, California.*

Determine all the prime numbers between one and two million for which $[N/2]+[N/2^2]+[N/2^3]+\cdots=N-3$ where the square brackets represent "the largest integer in" and the dots indicate that the process is to be carried as far as possible.

Solution by C. F. Pinzka, University of Cincinnati. Let $N=a_0+2a_1+2^2a_2+\cdots+2^sa_s$ be the binary representation of N. Then

$$\begin{aligned}
[N/2] &= a_1+2a_2+2^2a_3+\cdots+2^{s-1}a_s,\\
[N/2^2] &= a_2+2a_3+\cdots+2^{s-2}a_s,\\
[N/2^3] &= a_3+\cdots+2^{s-3}a_s,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
[N/2^s] &= a_s, \quad \text{and}\\
\sum_{r=1}^{s}[N/2^r] &= (2-1)a_1+(2^2-1)a_2+(2^3-1)a_3+\cdots+(2^s-1)a_s\\
&= N-\sum_{r=0}^{s}a_r = N-3.
\end{aligned}$$

Thus $\sum_{r=0}^{s}a_r=3$ and N is of the form $2^m+2^n+2^p$, where $m>n>p$. Since $10^6<N<2\cdot10^6$, $m=20$. Since N is prime, $p=1$, whence $1\leqq n\leqq 19$. Since $2\equiv-1 \pmod 3$, $2^{20}+2^n+1\equiv(-1)^n+2\not\equiv 0$ and it is clear that n cannot be even. Since $2^2\equiv-1 \pmod 5$, $2^{20}+2^n+1\equiv 2^n+2\not\equiv 0$, or $n\neq 3$, 7, 11, 15, 19. A similar test modulo 7 gives $n\neq 1$, 4, 7, 10, 13, 16, 19. Thus $n=5$, 9, or 17. A check with the RAND Corporation's *First Six Million Prime Numbers* (1957) shows that $2^{20}+2^5+1=1\ 048\ 609$, $2^{20}+2^9+1=1\ 049\ 089$, and $2^{20}+2^{17}+1=1\ 179\ 649$ are all primes. If the phrase "between one and two million" is interpreted as $1<N<2\cdot10^6$, the same techniques apply but the computations are somewhat tedious. For example, when $m=19$, the divisibility tests outlined above serve merely to eliminate $n=2$, 4, 5, 8, 11, 12, 14, 16, 17.

Also solved by Josef Andersson, Vaxholm, Sweden; Daniel I. A. Cohen, Central High School, Philadelphia, Pennyslvania; Gilbert Labelle, Collège de Longueuil, Canada; Prasert Na Nagara, College of Agriculture, Bangkok, Thailand; Robert E. Shafer, University of California, Berkeley; David L. Silverman, Beverley Hills, California; and the proposer.

Symmetric Conics

472 [January 1962]. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz., Turkey.*

Let (C) be a conic and M be a variable point on it. Let T be the point symmetric to M with respect to the main axis, and t the tangent line at T. Denote the intersection of the perpendicular from M to t with the line joining T to the center of the conic by I. If M' is symmetric to M with respect to I, prove that: 1. The locus of M' is another conic (C') of the same kind as (C). 2. The conics (C) and (C') are confocal.

Solution by R. D. H. Jones, College of William and Mary, Virginia. Let the conic be $x^2/a^2+y^2/b^2=1$, let M be $(a \cos \Delta,\ b \sin \Delta)$ so T is the point $(a \cos \Delta,\ -b \sin \Delta)$, and t, the tangent at T, is $(x/a) \cos \Delta-(y/b) \sin \Delta=1$. MI is the line through M perpendicular to t and therefore is:

$$y - b \sin \Delta = \frac{-a \sin \Delta}{b \cos \Delta} \cdot (x - a \cos \Delta). \tag{1}$$

The line joining T to the center of conic is

$$\frac{x}{a \cos \Delta} + \frac{y}{b \sin \Delta} = 0. \tag{2}$$

The point I is the intersection of (1) and (2) and is found to have coordinates:

$$\frac{a(a^2+b^2)}{a^2-b^2} \cos \Delta, \qquad \frac{-b(a^2+b^2)}{a^2-b^2} \sin \Delta.$$

By hypothesis M' is symmetric to M with respect to I and therefore has coordinates:

$$x_{M'} = 2x_I - x_M = \frac{a^3+3ab^2}{a^2-b^2} \cos \Delta$$

similarly

$$y_{M'} = 2y_I - y_M = \frac{-(3a^2, b+b^3)}{a^2-b^2} \sin \Delta.$$

Let

$$A = \frac{a(a^2+b^2)}{a^2-b^2} \quad \text{and} \quad B = \frac{b(a^2+b^2)}{a^2-b^2}.$$

If a and b are real and a greater than b, then A and B are real and A greater than B. Therefore if C is an ellipse the locus of M' is an ellipse. If, however, b is imaginary B is imaginary: hence if C is an hyperbola, so is the locus of M'. It is readily shown that $A^2-B^2=a^2-b^2$: therefore the locus of M' is confocal with C.

If C is a parabola we start de novo, letting M be $(at^2, 2at)$. T is $(at^2, -2at)$ and the tangent t is the line $y+2at=-(1/t)(x-at^2)$. Hence the line through M perpendicular to t is

$$y-2at=t(x-at^2) \tag{3}$$

The diameter through T is the line parallel to axis of parabola through T, that is $y=-2at$ and solving this equation with (3) we have the coordinates of I:

$$x_I=at^2-4a,\ y_I=-2at.$$

As before the coordinates of M' are given by:

$$x_{M'}=2x_I-x_M=at^2-8a,\ y_{M'}=2y_I-y_M=-6at.$$

Eliminating t we find M' lies on $x+8a=a(y/-6a)^2$. This is a parabola, and moving the origin to $(-8a, 0)$, it becomes $y^2=36ax$. The focus of this parabola is at $(9a, 0)$ i.e. $(a, 0)$ with respect to original axes. That is, the locus of M' is a parabola confocal with original parabola.

Also solved by Josef Andersson, Vaxholm, Sweden; Kit Hawes, Orange Coast College, California; Hazel S. Wilson, Jacksonville University, Florida; and the proposer.

A Quotient of Integrals

473 [January 1962]. *Proposed by Joseph W. Andrushkiw, Seton Hall University.*
Show that

$$\frac{\displaystyle\int_0^{2\pi}\frac{x^2\sin x\,dx}{1+\sin^2 x}}{\displaystyle\int_0^{2\pi}\frac{x\sin x\,dx}{1+\sin^2 x}}=2\pi.$$

Solution by Richard E. Sarber, Indiana Technical College. Starting with

$$\int_{-\pi}^{\pi}\frac{(y+\pi)^2\sin y\,dy}{1+\sin^2 y}$$

and letting

$$f(y)=\frac{\sin y}{1+\sin^2(y)}$$

yields

$$\int_{-\pi}^{\pi}(y+\pi)^2f(y)dy=\int_{-\pi}^{\pi}y^2f(y)dy+2\pi\int_{-\pi}^{\pi}yf(y)dy+\pi^2\int_{-\pi}^{\pi}f(y)dy$$

$$=2\pi\int_{-\pi}^{\pi}yf(y)dy+2\pi^2\int_{-\pi}^{\pi}f(y)dy$$

since $f(-y)=-f(y)$ and

$$\int_{-\pi}^{\pi} y^2 f(y)\,dy = \int_{-\pi}^{\pi} \pi^2 f(y)\,dy = \int_{-\pi}^{\pi} 2\pi^2 f(y)\,dy = 0$$

thus

$$\int_{-\pi}^{\pi} (y+\pi)^2 f(y)\,dy = 2\pi \int_{-\pi}^{\pi} (y+\pi) f(y)\,dy.$$

Now let $y = x - \pi$

$$\int_{0}^{2\pi} x^2 f(x-\pi)\,dx = 2\pi \int_{0}^{2\pi} x f(x-\pi)\,dx$$

since $f(x-\pi) = -f(x)$. Hence this yields

$$\frac{\displaystyle\int_{0}^{2\pi} x^2 f(x)\,dx}{\displaystyle\int_{0}^{2} x f(x)\,dx} = 2\pi.$$

Also solved by Josef Andersson, Vaxholm, Sweden; Maurice Brisebois, University of Sherbrooke, Canada; J. L. Brown, Jr., Pennsylvania State University (two solutions); L. Carlitz, Duke University; Henry E. Fettis, Wright Air Development Center, Ohio; Murray S. Klamkin, AVCO, Wilmington, Massachusetts; R. J. Mansfield, Research Council of Alberta, Canada; William R. Nico, Loyola University, Chicago, Illinois; C. F. Pinzka, University of Cincinnati; Sylvester Reese, Los Angeles, California; Robert E. Shafer, University of California, Berkeley; W. C. Waterhouse, Harvard University; Dale Woods, State Teachers College, Kirksville, Missouri; and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 299. Find the value of the expression $\cos 7° + \cos 127° - \cos 67°$. [*Submitted by J. M. Gandhi.*]

Q 300. Integrate

$$I = \int \frac{d\theta}{a + b\cos\theta}$$

without direct recourse to the usual substitution $z = \tan\theta/2$. [*Submitted by M. S. Klamkin.*]

Q 301. What are the necessary and sufficient conditions that the diagonals of Norman Anning's "crossed" plane quadrilateral (T 53) be parallel? [*Submitted by Charles E. Maley.*]

Q 302. Prove that $\binom{2n}{n}$ is even for all n. [*Submitted by Leo Moser.*]

Comments on Quickies

Q 241 [March 1959]. Show that the medians of a scalene triangle do not bisect any of the angles of the triangle.

A 241. *Alternate solution by C. W. Trigg.* An angle bisector divides the opposite side, c, into segments proportional to the adjacent sides, a and b. In a scalene triangle, $a \neq b$ so the division point is not the midpoint. Since the median is drawn to the midpoint it does not coincide with the angle bisector. This approach avoids the approximations inherent in the folding method of the original answer.

Q 295 [March 1962]. Determine the area of an ellipse with semi-major axis a and semi-minor axis b.

A 295. *Alternate solution by C. F. Pinzka.* Let $(x^2/a^2)+(y^2/b^2)=1$ be the equation of the ellipse. Then $x^2+y^2=a^2$ will be the circumscribing circle. The chords of the ellipse and circle which lie on any vertical line (cutting both curves) have the constant ratio b/a. Hence the area of the ellipse is $\pi a^2(b/a)=\pi ab$. This is essentially the same argument as Trigg's, but it does not involve 3 dimensions.

ANSWERS TO QUICKIES

A 299 $\cos A+\cos (120+A)-\cos (60+A)$
$=\cos A-\cos (60-A)-\cos (60+A)$
$=\cos A-2 \cos 60 \cos A$
$=\cos A-\cos A=0$. Therefore the required expression with $A=7°$ is zero.

A 300

$$I = \int \frac{d\theta}{(a-b)\sin^2\dfrac{\theta}{2} + (a+b)\cos^2\dfrac{\theta}{2}}$$

$$= 2\int \frac{d \tan \theta/2}{a+b+(a-b)\tan^2\theta/2}$$

$$= \frac{2}{\sqrt{a^2+b^2}} \text{arc tan} \sqrt{\frac{a-b}{a+b}} \tan \theta/2 \qquad \text{for } a > b \quad \text{or}$$

$$= \frac{2}{\sqrt{a^2+b^2}} \text{arc tanh} \sqrt{\frac{b-a}{b+a}} \tan \theta/2 \qquad \text{for } a < b.$$

A 301. The condition is that the areas of its two parts be equal. For then $ABD = CBD$, altitudes are equal and, hence, AC is parallel to BD. Conversely, if AC is parallel to BD, then $ABD=CBD$ and the areas of the two parts are equal.

A 302. We have $\binom{2n}{n}=2\binom{2n-2}{N}$ from Pascal's triangle.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The Association is a national organization of persons interested in mathematics at the college level. It was organized at Columbus, Ohio, in December 1915 with 1045 individual charter members and was incorporated in the State of Illinois on September 8, 1920. Its present membership is over 12,000, including more than 500 members residing in foreign countries.

Any person interested in the field of mathematics is eligible for election to membership. Annual dues of $5.00 includes a subscription to the American Mathematical Monthly. Members are also entitled to reduced rates for purchases of the Carus Mathematical Monographs, MAA Studies, and for subscriptions to several journals.

Further information about the Association, its publications and its activities may be obtained by writing to:

HARRY M. GEHMAN, *Executive Director*
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

MATHEMATICS MAGAZINE VOL. 35, NO. 4, SEPT.-OCT., 1962